NUMERICAL EVIDENCE FOR THE EQUIVARIANT BIRCH AND SWINNERTON-DYER CONJECTURE (PART II)

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Abstract. We continue the study of the Equivariant Tamagawa Number Conjecture for the base change of an elliptic curve begun in [1]. We recall that the methods developed in [1], apart from very special cases, cannot be applied to verify the l-part of the ETNC if l divides the order of the group. In this note we focus on extensions of l-power degree (l an odd prime) and describe methods for computing numerical evidence for ETNC$_l$. For cyclic l-power extensions we also express the validity of ETNC$_l$ in terms of explicit congruences.

1. Introduction

Let $E/\mathbb{Q}$ be an elliptic curve and $K/\mathbb{Q}$ a finite Galois extension with group G. We write E_K for the base change of E and consider the motive $M_K := h^1(E_K)(1)$ as a motive over $\mathbb{Q}$ with a natural action of the rational group ring $\mathbb{Q}[G]$.

In [1] we described an explicit formulation (under certain hypothesis) of the "Equivariant Tamagawa Number Conjecture" for the pair $(M_K, \mathbb{Z}[G])$ and developed algorithmic methods for computing numerical evidence. We recall that the "Equivariant Tamagawa Number Conjecture" is formulated in much greater generality. However, in this manuscript we will exclusively deal with the special case of the base change of an elliptic curve as above. The abbreviation ETNC will always refer to this case of the conjecture.

It is well known that the ETNC should be an equivariant form of the Birch and Swinnerton-Dyer conjecture (for short BSD). However, this is not obvious from the very general and comparatively abstract formulation of the ETNC in [11]. If $K = \mathbb{Q}$ (so no group is acting), the equivalence of the two conjectures is shown in [20] or [29]. For arbitrary Galois extensions $K/\mathbb{Q}$ one can make use of the notion of refined Euler characteristics introduced in [12] in order to formulate the ETNC as an explicit equality in a relative algebraic K-group which makes the relation to the BSD conjecture transparent. This is the main theoretical result of the manuscript [1], see in particular Proposition 4.4 of loc.cit. However, we had to impose quite strong hypothesis in order to derive this result. The aim of this paper is to relax part of these hypothesis.

Assuming that the Mordell-Weil group $E(K)$ or a subgroup of finite index is known, we showed in loc.cit. how to compute numerical evidence for the rationality part of the ETNC (see Remark 4.1 for more details). We further described how one can use these computations to numerically verify the l-part of the ETNC for all primes l outside a finite set of difficult primes. This finite set contains in most

Date: April 26, 2011.

cases the prime divisors of $\#G$ and $\#Ш(E/K)$ (which we always assume to be finite!). To explain the reason why we had to exclude prime divisors of $\#G$ we recall that the approach of loc.cit. is restricted to the case that certain groups (e.g. the Mordell-Weil group $E(K)\otimes_{\mathbb{Z}}\mathbb{Z}_l$), which occur as cohomology modules of naturally defined perfect complexes, are $\mathbb{Z}_l[G]$-perfect. If $l\nmid\#G$ this condition is always fulfilled because $\mathbb{Z}_l[G]$ is a regular ring. However, if $l\mid\#G$, then there are only some very rare cases where the cohomology modules under consideration are perfect.

In this note we focus on the l-part of the ETNC (for short ETNC_l) for primes l dividing $\#G$. Our approach is motivated by and to a large extend based on work of Burns in [13].

We briefly recall the definition of an 'augmented trivialized extension' (for short a.t.e.) from [13, Sec. 3]. Let Λ be a Dedekind domain of characteristic zero and quotient field F. We fix an extension field $\mathbb{E}$ of F, a finite dimensional semisimple F-algebra A and a Λ-order $\mathcal{A}$ in A. Then an augmented $\mathbb{E}$-trivialized extension of $\mathcal{A}$-modules is a triple $\tau=(\varepsilon_\tau,\lambda_\tau,\mathcal{L}_\tau^*)$ consisting of a perfect 2-extension $\varepsilon_\tau\in\mathrm{Ext}^2_{\mathcal{A}}(H^1_\tau,H^0_\tau)$ of finitely generated $\mathcal{A}$-modules H^0_τ and H^1_τ, an isomorphism $\lambda_\tau:\mathbb{E}\otimes_\Lambda H^0_\tau\longrightarrow\mathbb{E}\otimes_\Lambda H^1_\tau$ of $(\mathbb{E}\otimes_F A)$-modules and an element $\mathcal{L}_\tau^*$ in the center of $\mathbb{E}\otimes_F A$. Associated to an a.t.e. τ Burns defined in [13, Sec. 3] an Euler characteristic $\chi_{\mathcal{A},\mathbb{E}}(\tau)\in K_0(\mathcal{A},\mathbb{E})$. We recall the explicit definition in Section 3.

We need to introduce some further notations. For a ring R we write $\zeta(R)$ for its center. Let l be a prime and let $\mathbb{C}_l$ denote the completion of a fixed algebraic closure of $\mathbb{Q}_l$. For a finite group G we write $\mathrm{Irr}(G)$ for the set of absolutely irreducible characters and $\mathrm{Irr}_{\mathbb{Q}}(G)$ for a set of representatives of $G_{\mathbb{Q}}$-orbits of $\mathrm{Irr}(G)$ under the action of the absolute Galois group $G_{\mathbb{Q}}$. As usual we write d_K for the discriminant of K and N_E for the conductor of E. For a prime p we write $c_p(E)$ for the usual Tamagawa factor and if $p\nmid N_E$ we let $\bar{E}(\mathbb{F}_p)$ denote the group of $\mathbb{F}_p$-rational points on the reduced curve E modulo p.

In our applications we look for elliptic curves $E/\mathbb{Q}$ and Galois extensions $K/\mathbb{Q}$ such that the validity of ETNC_l can be decided by considering an augmented trivialized extension where $\Lambda=\mathbb{Z}_l,\mathbb{E}=\mathbb{C}_l,A=\mathbb{Q}_l[G],\mathcal{A}=\mathbb{Z}_l[G]$ and $H^0_\tau=\mathbb{Z}_l\otimes_{\mathbb{Z}}E(K),H^1_\tau=\mathrm{Hom}_{\mathbb{Z}_l}(\mathbb{Z}_l\otimes_{\mathbb{Z}}E(K),\mathbb{Z}_l)=:(\mathbb{Z}_l\otimes_{\mathbb{Z}}E(K))^*$, the trivialization $\lambda_\tau:=\lambda_{NT}$ is induced by the Néron-Tate height pairing and, finally, $\mathcal{L}_\tau^*$ is the leading term $L^*(M_K)$ of the equivariant motivic L-function at $s=0$.

We impose the following hypothesis which essentially coincides with condition (B) of [13].

Hypothesis:

(i) $[K:\mathbb{Q}]=l^n, l$ an odd prime,
(ii) $(d_K,l)=1,\quad(d_K,N_E)=1$,
(iii) $l\nmid\#E(\mathbb{Q})_{tors}\prod_{p\mid d_K}\#\bar{E}(\mathbb{F}_p)$,
(iv) $l\nmid N_E$,
(v) $l\nmid\prod_{p\mid N_E}c_p(E)$,
(vi) $l\nmid\#Ш(E/K)$.

By Wedderburns' theorem we canonically identify $\zeta(\mathbb{C}[G])$ and $\bigoplus_{\psi\in\mathrm{Irr}(G)}\mathbb{C}$. We write $R=(R_\psi)_{\psi\in\mathrm{Irr}(G)}$ for the vector of resolvents and $\Omega=(\Omega_\psi)_{\psi\in\mathrm{Irr}(G)}$ for the vector of periods. For a precise definition we refer the reader to [1, Prop. 3.1]. Let

$$\delta_l:\mathbb{C}_l[G]^\times/\mathrm{Nrd}_{\mathbb{Q}_l[G]}(K_1(\mathbb{Z}_l[G]))\longrightarrow K_0(\mathbb{Z}_l[G],\mathbb{C}_l)$$

denote the canonical isomorphism (see e.g. [5, Th. 2.3(ii)]). Then $\delta_l(\Omega R^{-1})$ should be regarded as the equivariant period. We set

$$\xi_l := \prod_{p|d_K} (L_p(E,\bar{\chi},1))^{-1}_{\chi\in\mathrm{Irr}(G)}$$

where $L_p(E,\chi,1)$ denotes the local Euler factor at $s=1$.

Theorem 1.1. *Assume (i)-(vi). Then there exists an augmented trivialized extension $\tau_0 = (\varepsilon_{\tau_0}, \lambda_{NT}, L^*(M_K))$ of $H^0_{\tau_0} = \mathbb{Z}_l \otimes_{\mathbb{Z}} E(K)$ and $H^1_{\tau_0} = (\mathbb{Z}_l \otimes_{\mathbb{Z}} E(K))^*$, such that ETNC$_l$ holds if and only if*

$$\chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(\tau_0) + \delta_l(\Omega R^{-1}) + \delta_l(\xi_l) = 0$$

in $K_0(\mathbb{Z}_l[G],\mathbb{C}_l)$.

The extension class $\varepsilon_{\tau_0} \in \mathrm{Ext}^2_{\mathbb{Z}_l[G]}((\mathbb{Z}_l\otimes_{\mathbb{Z}} E(K))^*, \mathbb{Z}_l\otimes_{\mathbb{Z}} E(K))$ is specified by the ETNC, however, it is not clear how to use this information for explicit numerical experiments.

We therefore adopt the following strategy in this paper. We calculate the Ext-group

$$\mathcal{E}xt := \mathrm{Ext}^2_{\mathbb{Z}_l[G]}((\mathbb{Z}_l \otimes_{\mathbb{Z}} E(K))^*, \mathbb{Z}_l \otimes_{\mathbb{Z}} E(K))$$

and compute for each perfect extension class $\varepsilon \in \mathcal{E}xt$ the refined Euler characteristic $\chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(\tau(\varepsilon))$ associated to $\tau(\varepsilon) = (\varepsilon, \lambda_{NT}, L^*(M_K))$. In this way we obtain an explicit subset

$$\mathcal{C} := \{\chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(\tau(\varepsilon)) + \delta_l(\Omega R^{-1}) + \delta_l(\xi_l) \mid \varepsilon \in \mathcal{E}xt \text{ perfect } \}$$

of $K_0(\mathbb{Z}_l[G],\mathbb{C}_l)$. The validity of ETNC$_l$ now predicts that $\mathcal{C}$ is contained in $K_0(\mathbb{Z}_l[G],\mathbb{Q}_l)_{tors}$ and contains the trivial element. This can be numerically verified by the methods of [5].

If $\mathrm{rk}(E(K)) = 0$ (and $l \nmid E(K)_{tors}$ by (iii)), then $\mathcal{E}xt$ is trivial and we obtain

Corollary 1.2. *Assume (i)-(vi) and, in addition, that $\mathrm{rk}(E(K)) = 0$. Then ETNC$_l$ holds if and only if*

$$\delta_l(\Omega R^{-1}) + \delta_l(\xi_l) = \delta_l(L^*(M_K)).$$

in $K_0(\mathbb{Z}_l[G],\mathbb{C}_l)$.

We can use the corollary to numerically verify the full ETNC$_l$ (provided that we assume the rationality conjecture). In Section 4, see in particular (7), (8) and (9), we very explicitly describe what has to be checked.

Of course, in the case $\mathrm{rk}(E(K)) > 0$ it would be very interesting to describe the 'correct' extension class ε_{τ_0} theoretically, but explicitly enough, so that the information can be used to fully verify the l-part of the ETNC numerically. This seems to be an interesting and difficult problem which we do not touch in this manuscript. Instead, in order to push our general approach a little further in a special case we impose in addition to (i)-(vi) the

Hypothesis:

(vii) $G = \langle g_0 \rangle$ is cyclic of order l^n, $l \neq 2$ prime, $n > 0$,
(viii) $\mathrm{rk}_{\mathbb{Z}} E(K) = \mathrm{rk}_{\mathbb{Z}} E(\mathbb{Q})$,

In Definition (4.4) of Section 4 we define an explicit subgroup $\mathcal{E}$ of $K_0(\mathbb{Z}_l[G],\mathbb{Q}_l)_{tors}$ of order $(l-1)n^{n-1}$.

Theorem 1.3. *Assume (i) - (viii). Then ETNC$_l$ is valid modulo $\mathcal{E}$ if and only if $\mathcal{E} = \mathcal{C}$*

For a more precise statement see Theorem 4.5. Note that by Proposition 5.4 we have

$$l^{n-1}(l-1) = \#\mathcal{E} \ll \#K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors} = (l-1)^n l^e \text{ with } e = \frac{l^n - 1}{l-1} - n.$$

Again, by the methods developed in [5], we can use Theorem 1.3 for numerical computations. We give a very explicit description of what has to be checked at the end of Section 4.

In Section 5 we will express the validity of ETNC$_l$ for cyclic groups of prime power order in terms of explicit congruences (see Propostion 5.2).

We recall that ETNC$_l$ modulo $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors}$ is equivalent to ETNC$_l$ for the pair $(M_K, \mathcal{M})$ where $\mathcal{M}$ is a maximal order such that $\mathbb{Z}[G] \subseteq \mathcal{M} \subseteq \mathbb{Q}[G]$. Studying the ETNC in this case is much easier (but still very difficult) because $\mathcal{M}$ is regular so that we do not have to consider questions of perfectness of modules, and consequently, also do not have to use any extension class information. In return, ETNC$_l$ for the pair $(M_K, \mathcal{M})$ will not imply any of the fine explicit congruences predicted by ETNC$_l$ for the pair $(M_K, \mathbb{Z}[G])$.

As in [1] this manuscript mainly deals with the additional difficulties and consequences of equivariant integrality conjectures. We therefore usually assume the rationality conjecture. We recall, however, that there are important results in the literature (without being exhaustive we only mention [18, 21, 22, 23, 30] and recent results of Bertolini and Darmon) from which one can possibly deduce the equivariant rationality conjecture provided that the analytic (equivariant) rank is at most 1. Furthermore we throughout assume that the Tate-Shafarevic group $Ш(E/K)$ is finite. Again it is possible to deduce finiteness of $Ш(E/K)$ in many examples provided that the analytic rank is at most 1 from the above mentioned work.

The structure of this paper is as follows. In Section 2 we describe an algorithm for the computation of Ext-groups $\mathrm{Ext}^2_{\mathbb{Z}[G]}(H^1, H^0)$ where H^0, H^1 denote finitely generated $\mathbb{Z}[G]$-modules. In Section 3 we show how to compute the refined Euler characteristic associated to augmented trivialized extensions $\tau = (\varepsilon, \lambda, \mathcal{L}^*)$ where $\lambda : \mathbb{R} \otimes_{\mathbb{Z}} H^0 \xrightarrow{\simeq} \mathbb{R} \otimes_{\mathbb{Z}} H^1$ is an isomorphism of $\mathbb{R}[G]$-modules, $\mathcal{L}^*$ an element in the center of $\mathbb{R}[G]$ and $\varepsilon \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(H^1, H^0)$. By localization we obtain the set $\mathcal{C}$ from above. In Section 4 we focus on extensions of l-power degree and prove Theorems 1.1 and 1.3 and the corollary. Section 5 is dedicated to the study of explicit congruences which are implied by the triviality of elements of $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$ for cyclic groups of prime power order l^n. We also compute the order of the torsion subgroup. Finally, in Section 6 we give a brief account of our numerical experiments.

Notations: For a commutative ring Λ and a Λ-module M we write M^* for the Λ-linear dual $\mathrm{Hom}_\Lambda(M, \Lambda)$. For a $\mathbb{Z}_l$-module W we write $W^\vee$ for the Pontriyagin dual $\mathrm{Hom}_{cont}(W, \mathbb{Q}_l/\mathbb{Z}_l)$. If $\mathbb{E}/\Lambda$ is an extension of commutative rings and M a Λ-module, then we often set $M_{\mathbb{E}} := M \otimes_\Lambda \mathbb{E}$.

Acknowledgments: I would like to thank David Burns for valuable discussions and the anonymous referees for many helpful comments and suggestions.

2. Computation of Ext-groups

Let G be a finite group and X, Y finitely generated $\mathbb{Z}[G]$-modules. In this section we describe an algorithm which computes $\mathrm{Ext}^n_{\mathbb{Z}[G]}(X,Y), n \geq 1$, as an abstract finitely generated abelian group.

We assume that X and Y are given in the form

$$\begin{aligned} X &= \mathbb{Z}x_{t,1} \oplus \ldots \oplus \mathbb{Z}x_{t,m} \oplus \mathbb{Z}x_{tf,1} \oplus \ldots \oplus \mathbb{Z}x_{tf,n} \\ Y &= \mathbb{Z}y_{t,1} \oplus \ldots \oplus \mathbb{Z}y_{t,r} \oplus \mathbb{Z}y_{tf,1} \oplus \ldots \oplus \mathbb{Z}y_{tf,s} \end{aligned}$$

with $\mathbb{Z}$-free generators $x_{tf,1}, \ldots, x_{tf,n}, y_{tf,1}, \ldots, y_{tf,s}$ and torsion elements $x_{t,1}, \ldots, x_{t,m}, y_{t,1}, \ldots, y_{t,r}$.

In addition, we assume that the G-action on these generators is explicitly computable. It is then straightforward to write down a naive algorithm which computes a short exact sequence

$$0 \longrightarrow R \longrightarrow \mathbb{Z}[G]^k \longrightarrow X \longrightarrow 0 \tag{1}$$

for any $\mathbb{Z}[G]$-module X given in the above form. Furthermore, R can again be described by a set of $\mathbb{Z}$-generators with explicit G-action. Hence we can compute n-syzygies of modules X as above for all $n \geq 1$. Note, however, that it is not clear how to compute a presentation of the form (1) with small or even minimal k. We will not discuss this problem in this manuscript.

We compute once and for all an n-syzygy

$$0 \longrightarrow Z \xrightarrow{\iota} F^0 \longrightarrow F^1 \longrightarrow \cdots \longrightarrow F^{n-1} \longrightarrow X \longrightarrow 0$$

with finitely generated free $\mathbb{Z}[G]$-modules $F^0, \ldots, F^{n-1}$ and a finitely generated $\mathbb{Z}[G]$-module Z. Then, e.g. by [14, (8.3)],

$$\mathrm{Ext}^n_{\mathbb{Z}[G]}(X,Y) \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(Z,Y)/\iota^*\left(\mathrm{Hom}_{\mathbb{Z}[G]}(F^0,Y)\right)$$

and we will explicitly describe how the right hand side can be computed as a finitely generated abelian group.

We compute a resolution

$$0 \longrightarrow R \longrightarrow \mathbb{Z}[G]^k \longrightarrow Z \longrightarrow 0$$

and let R_0 denote a finite $\mathbb{Z}$-generating set for R. Each $\varphi \in \mathrm{Hom}_{\mathbb{Z}[G]}(Z,Y)$ is uniquely determined by the images $y_i := \varphi(w_i + R)$, $i = 1, \ldots, k$, where we write $w_1, \ldots, w_k$ for the canonical $\mathbb{Z}[G]$-basis of $\mathbb{Z}[G]^k$. Conversely, a set $\{y_1, \ldots, y_k\}$ determines a map φ, if and only if $\varphi(\rho) = 0$ for all $\rho \in R_0$. If we set

$$\rho = \sum_{i=1}^{k} \lambda_i^{(\rho)} w_i \text{ with } \lambda_i^{(\rho)} \in \mathbb{Z}[G],$$

then

$$\varphi(\rho) = \varphi\left(\sum_{i=1}^{k} \lambda_i^{(\rho)} w_i\right) = \sum_{i=1}^{k} \lambda_i^{(\rho)} y_i.$$

Let $y_i = \sum_{j=1}^{r} a_{ij} y_{t,j} + \sum_{j=1}^{s} b_{ij} y_{tf,j}$ with $a_{ij}, b_{ij} \in \mathbb{Z}$. Note that the a_{ij} are only determined modulo $\mathrm{ord}(y_{t,j})$ by y_i. Then

$$\begin{aligned}
\varphi(\rho) &= \sum_{i=1}^{k} \lambda_i^{(\rho)} y_i = \sum_{i=1}^{k} \left(\sum_{j=1}^{r} a_{ij} \lambda_i^{(\rho)} y_{t,j} + \sum_{j=1}^{s} b_{ij} \lambda_i^{(\rho)} y_{tf,j} \right) \\
&= \sum_{i=1}^{k} \left(\sum_{j=1}^{r} a_{ij} \sum_{q=1}^{r} c_{ijq}^{(\rho)} y_{t,q} + \sum_{j=1}^{s} b_{ij} \left(\sum_{q=1}^{r} d_{ijq}^{(\rho)} y_{t,q} + \sum_{q=1}^{s} e_{ijq}^{(\rho)} y_{tf,q} \right) \right) \\
&= \sum_{q=1}^{r} \left(\sum_{i=1}^{k} \sum_{j=1}^{r} a_{ij} c_{ijq}^{(\rho)} + \sum_{i=1}^{k} \sum_{j=1}^{s} b_{ij} d_{ijq}^{(\rho)} \right) y_{t,q} + \\
&\quad \sum_{q=1}^{s} \left(\sum_{i=1}^{k} \sum_{j=1}^{s} b_{ij} e_{ijq}^{(\rho)} \right) y_{tf,q}
\end{aligned}$$

where for $\rho \in R_0$ and $i = 1, \ldots, k$,

$$\begin{aligned}
\lambda_i^{(\rho)} y_{t,j} &= \sum_{q=1}^{r} c_{ijq}^{(\rho)} y_{t,q}, \\
\lambda_i^{(\rho)} y_{tf,j} &= \sum_{q=1}^{r} d_{ijq}^{(\rho)} y_{t,q} + \sum_{q=1}^{s} e_{ijq}^{(\rho)} y_{tf,q}.
\end{aligned}$$

Equating coefficients we see that $\varphi(\rho) = 0$ is equivalent to the system of linear congruences and equations

$$\sum_{i=1}^{k} \sum_{j=1}^{r} a_{ij} c_{ijq}^{(\rho)} + \sum_{i=1}^{k} \sum_{j=1}^{s} b_{ij} d_{ijq}^{(\rho)} \equiv 0 (\mathrm{mod}\ \mathrm{ord}(y_{t,q})), \quad 1 \le q \le r, \rho \in R_0,$$

$$\sum_{i=1}^{k} \sum_{j=1}^{s} b_{ij} e_{ijq}^{(\rho)} = 0, \quad 1 \le q \le s, \rho \in R_0.$$

Let $\mathcal{W}$ denote the set of solutions for the vectors $(a_{ij}, b_{ij})^t \in \mathbb{Z}^{kr+ks}$. Then $\mathcal{W}$ is of the form

$$\mathcal{W} = \mathbb{Z} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \oplus \ldots \oplus \mathbb{Z} \begin{pmatrix} a_m \\ b_m \end{pmatrix}, \quad m \ge 0, \quad a_l \in \mathbb{Z}^{kr}, b_l \in \mathbb{Z}^{ks}, l = 1, \ldots, m.$$

Consider $\mathcal{W}_0 = \langle (e_{ij}, 0)^t, 1 \le i \le k, 1 \le j \le r \rangle_{\mathbb{Z}}$ with

$$e_{ij} = (\underbrace{0, \ldots, 0}, \ldots, \underbrace{0, \ldots, 0, m_j, 0, \ldots, 0}_{i-\text{th component}}, \ldots, \underbrace{0, \ldots, 0})^t \in \mathbb{Z}^{kr},$$

where the only non zero entry is in the i-th component and is given by $m_j = \mathrm{ord}(y_{t,j})$ at the j-th position. Then $(e_{ij}, 0)^t$ corresponds to $y_i = 0$, so clearly $\mathcal{W}_0 \subseteq \mathcal{W}$. The quotient $\mathcal{W}/\mathcal{W}_0$, which can be computed by [15, Alg. 4.1.7], is easily shown to represent $\mathrm{Hom}_{\mathbb{Z}[G]}(Z, Y)$.

Finally we have to compute the submodule $\iota^*(\mathrm{Hom}_{\mathbb{Z}[G]}(F^0, Y))$. Since F^0 is a free $\mathbb{Z}[G]$-module, say $F^0 \simeq \mathbb{Z}[G]^l$, we obtain $\mathrm{Hom}_{\mathbb{Z}[G]}(F^0, Y)) \simeq Y^l$. So we may assume that $\mathrm{Hom}_{\mathbb{Z}[G]}(F^0, Y))$ is given by a finite set of $\mathbb{Z}$-generators Ψ. Then for each $\psi \in \Psi$ the homomorphism $\iota^*(\psi) = \psi \circ \iota$ can be identified with an element of

$\mathcal{W}/\mathcal{W}_0$ (by computing discrete logarithms as described after [15, Alg. 4.1.7]) and again using [15, Alg. 4.1.7] we determine the quotient $(\mathcal{W}/\mathcal{W}_0)/\langle \iota^*(\Psi)\rangle$ which by construction is isomorphic to $\mathrm{Ext}^n_{\mathbb{Z}[G]}(X,Y)$.

3. Computation of refined Euler characteristics

The main references for this section are [12] and [13, Sec. 3]. For the convenience of the reader we recall the relevant definitions and results.

Let Λ be either $\mathbb{Z}$ or $\mathbb{Z}_l$ and G as before a finite group. A $\Lambda[G]$-module M is called perfect if the associated complex $M[0]$ is quasi-isomorphic to a bounded complex of finitely generated $\Lambda[G]$-projective modules. By [13, Sec. 2.6] the following conditions are equivalent for a finitely generated $\Lambda[G]$-module M:

- *a)* M is perfect.
- *b)* M is cohomologically trivial.
- *c)* M is of finite projective dimension (as a $\Lambda[G]-$module).

The proof relies on [26, Th. 1, Prop. 4 and Remark, page 175]. There it is shown that a finitely generated Λ-torsion-free Λ-module is $\Lambda[G]$-projective, if and only if it is G-cohomologically trivial (which we abbreviate by c.t. in the following). More precisely, it follows that any finitely generated c.t. $\Lambda[G]$-module is of projective dimension at most 2.

Let H^0, H^1 be finitely generated $\Lambda[G]$-modules. An extension class $\varepsilon \in \mathrm{Ext}^2_{\Lambda[G]}(H^1,H^0)$ is said to be perfect, if it can be represented as a Yoneda extension by an exact sequence

$$0 \longrightarrow H^0 \longrightarrow M^0 \xrightarrow{f} M^1 \longrightarrow H^1 \longrightarrow 0$$

with perfect $\Lambda[G]$-modules M^0 and M^1. Without loss of generality we can assume that M^1 is finitely generated $\Lambda[G]$-projective and M^0 finitely generated and of finite projective dimension as a $\Lambda[G]$-module.

We write F for the quotient field of Λ and let $\mathbb{E}$ be an extension field of F. If $\varepsilon \in \mathrm{Ext}^2_{\Lambda[G]}(H^1,H^0)$ is perfect and $\lambda\colon \mathbb{E}\otimes_\Lambda H^0 \longrightarrow \mathbb{E}\otimes_\Lambda H^1$ an isomorphism of $\mathbb{E}[G]$-modules then the refined Euler characteristic $\chi_{\Lambda[G],\mathbb{E}}(\varepsilon,\lambda) \in K_0(\Lambda[G],\mathbb{E})$ is defined in the following manner. We fix a projective resolution

$$0 \longrightarrow Q \xrightarrow{\iota} P \xrightarrow{\pi} M^0 \longrightarrow 0$$

of M^0. Then we have a quasi-isomorphism

$$\begin{array}{ccccc} Q & \xrightarrow{\iota} & P & \xrightarrow{d:=f\circ\pi} & M^1 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle =} \\ & & M^0 & \xrightarrow{f} & M^1 \end{array} \tag{2}$$

of complexes (centered in degrees $-1, 0$ and 1) and short exact sequences

$$\begin{aligned} &0 \longrightarrow \ker(d) \longrightarrow P \xrightarrow{d} \mathrm{im}(d) \longrightarrow 0,\\ &0 \longrightarrow Q \xrightarrow{\iota} \ker(d) \longrightarrow H^0 \longrightarrow 0,\\ &0 \longrightarrow \mathrm{im}(d) \longrightarrow M^1 \longrightarrow H^1 \longrightarrow 0. \end{aligned}$$

Tensoring with $\mathbb{E}$ (denoted by a subscript $\mathbb{E}$) and choosing splittings we obtain an isomorphism λ_{triv} of $\mathbb{E}[G]$-modules as the following composite

$$\begin{array}{rcl} P_{\mathbb{E}} & \longrightarrow & \ker(d)_{\mathbb{E}} \oplus \mathrm{im}(d)_{\mathbb{E}} \longrightarrow Q_{\mathbb{E}} \oplus H^0_{\mathbb{E}} \oplus \mathrm{im}(d)_{\mathbb{E}} \\ & \stackrel{(id,\lambda,\mathrm{id})}{\longrightarrow} & Q_{\mathbb{E}} \oplus H^1_{\mathbb{E}} \oplus \mathrm{im}(d)_{\mathbb{E}} \longrightarrow Q_{\mathbb{E}} \oplus M^1_{\mathbb{E}}. \end{array}$$

Then

$$\chi_{\Lambda[G],\mathbb{E}}(\varepsilon,\lambda) := [P, \lambda_{triv}, Q \oplus M^1] \in K_0(\Lambda[G],\mathbb{E}).$$

By the results of [7] and [12] this construction is independent of all choices.

Finally, an augmented $\mathbb{E}$-trivialized extension of $\Lambda[G]$-modules is a triple $\tau = (\varepsilon_\tau, \lambda_\tau, \mathcal{L}^*_\tau)$ comprising a perfect 2-extension $\varepsilon_\tau \in \mathrm{Ext}^2_{\Lambda[G]}(H^1_\tau, H^0_\tau)$ of finitely generated $\Lambda[G]$-modules, an isomorphism $\lambda_\tau \colon H^0_{\tau,\mathbb{E}} \longrightarrow H^1_{\tau,\mathbb{E}}$ of $\mathbb{E}[G]$-modules and an element $\mathcal{L}^*_\tau \in \zeta(\mathbb{E}[G])^\times$. We define the refined Euler characteristic of τ by

$$\chi_{\Lambda[G],\mathbb{E}}(\tau) := \chi_{\Lambda[G],\mathbb{E}}(\varepsilon_\tau, \lambda_\tau) - \delta_\Lambda(\mathcal{L}^*_\tau) \in K_0(\Lambda[G],\mathbb{E})$$

where $\delta_\Lambda \colon \zeta(\mathbb{E}[G])^\times \longrightarrow K_0(\Lambda[G],\mathbb{E})$ denotes the extended boundary homomorphism of [8, Lemma 2.1] or [11, Sec. 4.2].

Let now H^0 and H^1 be finitely generated $\mathbb{Z}[G]$-modules, $\lambda : \mathbb{R}\otimes_{\mathbb{Z}} H^0 \longrightarrow \mathbb{R}\otimes_{\mathbb{Z}} H^1$ an $\mathbb{R}[G]$-isomorphism and $\mathcal{L}^* \in \zeta(\mathbb{R}[G])$. Then, for each perfect extension class $\varepsilon \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(H^1, H^0)$, the triple $\tau_\varepsilon = (\varepsilon, \lambda, \mathcal{L}^*)$ is an augmented $\mathbb{R}$-trivialized extension of $\mathbb{Z}[G]$-modules. Our aim is to combine the computation of $\mathrm{Ext}^2_{\mathbb{Z}[G]}(H^1, H^0)$ with methodes developed in [5] to compute the set

$$\mathcal{C} = \mathcal{C}(H^0, H^1, \lambda, \mathcal{L}^*) := \left\{ \chi_{\mathbb{Z}[G],\mathbb{R}}(\tau_\varepsilon) \mid \varepsilon \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(H^1, H^0) \text{ perfect} \right\} \subseteq K_0(\mathbb{Z}[G],\mathbb{R}).$$

Actually, when considering the integrality part of the ETNC we will work prime by prime. Let l be a prime and let $\mathbb{C}_l$ denote the completion of a fixed algebraic closure of $\mathbb{Q}_l$. For every embedding $j : \mathbb{R} \longrightarrow \mathbb{C}_l$ we obtain induced maps $j_* : K_0(\mathbb{Z}[G],\mathbb{R}) \longrightarrow K_0(\mathbb{Z}_l[G],\mathbb{C}_l)$ and $j_* : \zeta(\mathbb{R}[G]) \longrightarrow \zeta(\mathbb{C}_l[G])$. We are then interested in the set $\mathcal{C}_{l,j} := j_*(\mathcal{C})$.

We first recall that for finitely generated $\mathbb{Z}[G]$-modules H^0 and H^1 one has

$$\mathbb{Z}_l \otimes_{\mathbb{Z}} \mathrm{Ext}^2_{\mathbb{Z}[G]}(H^1, H^0) \simeq \mathrm{Ext}^2_{\mathbb{Z}_l[G]}(\mathbb{Z}_l \otimes_{\mathbb{Z}} H^1, \mathbb{Z}_l \otimes_{\mathbb{Z}} H^0). \tag{3}$$

It follows that

$$\mathcal{C}_{l,j} = \left\{ \chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(\varepsilon_l, \lambda_{l,j}, j_*(\mathcal{L}^*)) \mid \varepsilon_l \in \mathrm{Ext}^2_{\mathbb{Z}_l[G]}(\mathbb{Z}_l \otimes_{\mathbb{Z}} H^1, \mathbb{Z}_l \otimes_{\mathbb{Z}} H^0) \text{ perfect} \right\}$$

where $\lambda_{l,j} \colon \mathbb{C}_l \otimes_{\mathbb{Z}} H^0 \longrightarrow \mathbb{C}_l \otimes_{\mathbb{Z}} H^1$ is induced by λ and the canonical isomorphisms $\mathbb{C}_l \otimes_{\mathbb{R},j} (\mathbb{R} \otimes_{\mathbb{Z}} H^i) \simeq \mathbb{C}_l \otimes_{\mathbb{Z}} H^i$ for $i = 0, 1$.

By (3) we can use the computational approach of Section 2 in order to compute $\mathrm{Ext}^2_{\mathbb{Z}_l[G]}(\mathbb{Z}_l \otimes_{\mathbb{Z}} H^1, \mathbb{Z}_l \otimes_{\mathbb{Z}} H^0)$. In the following we therefore assume that $\Lambda = \mathbb{Z}$ or $\mathbb{Z}_l$, H^0 and H^1 are finitely generated $\Lambda[G]$-modules and either $\lambda : H^0_{\mathbb{R}} \longrightarrow H^1_{\mathbb{R}}$ or $\lambda : H^0_{\mathbb{C}_l} \longrightarrow H^1_{\mathbb{C}_l}$.

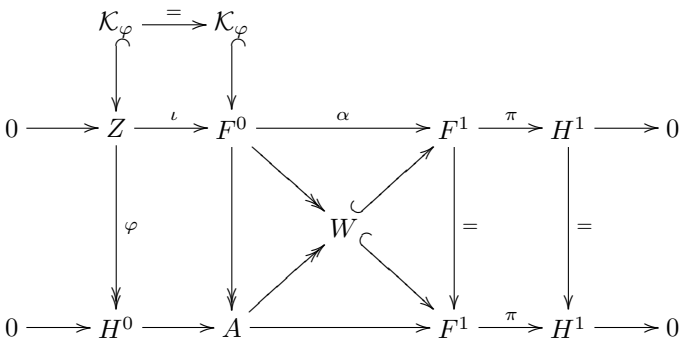
We fix a 2-syzygy

$$0 \longrightarrow Z \stackrel{\iota}{\longrightarrow} F^0 \stackrel{\alpha}{\longrightarrow} F^1 \stackrel{\pi}{\longrightarrow} H^1 \longrightarrow 0$$

and compute $\mathrm{Ext}^2_{\Lambda[G]}(H^1, H^0)$ as described in Section 2. So each element $\varepsilon \in \mathrm{Ext}^2_{\Lambda[G]}(H^1, H^0)$ is represented by a $\Lambda[G]$-homomorphism $\varphi : Z \longrightarrow H^0$. Without

loss of generality we may assume that φ is onto. We set $\mathcal{K}_\varphi := \ker(\varphi)$ and obtain a diagram of the form

(4)



with $W := \ker(\pi)$ and A denoting the pushout along ι and φ. We write $\varepsilon(\varphi) \in \mathrm{Ext}^2_{\Lambda[G]}(H^1, H^0)$ for the 2-extension represented by the bottom exact sequence.

Lemma 3.1. *$\varepsilon(\varphi)$ is a perfect 2-extension, if and only if $\mathcal{K}_\varphi$ is $\Lambda[G]$-projective (or equivalently, G-c.t.).*

Proof. If $\mathcal{K}_\varphi$ is c.t., then the long exact sequence of Galois cohomology implies that A is also c.t.. Conversely, if $\varepsilon(\varphi)$ is perfect, then $\varepsilon(\varphi)$ induces an isomorphism $H^i(G, H^1) \longrightarrow H^{i+2}(G, H^0)$ in all degrees of cohomology. This implies that A is c.t. and again from the long exact sequence of Galois cohomology we deduce the cohomological triviality of $\mathcal{K}_\varphi$. □

In order to select the extension classes which are perfect we need a method to decide whether $\mathcal{K}_\varphi$ is $\Lambda[G]$-projective. From our algorithm in Section 2 we obtain $\mathbb{Z}[G]$-homomorphisms $\varphi : Z \longrightarrow H^0$ representing the extension classes of $\mathrm{Ext}^2_{\mathbb{Z}[G]}(H^1, H^0)$. Hence we can compute the $\mathbb{Z}[G]$-module $\mathcal{K}_\varphi$. We recall from [14, Theorem (32.11)] that a finitely generated $\mathbb{Z}[G]$-module is projective, if and only if it is locally free. Similary, if $l \mid \#G$, then the $\mathbb{Z}_l[G]$-module $\mathbb{Z}_l \otimes_{\mathbb{Z}} \mathcal{K}_\varphi$ is projective, if and only if it is $\mathbb{Z}_l[G]$-free. This follows from [14, Theorem (32.1)] together with the fact that $\mathbb{Q}_l \otimes_{\mathbb{Z}} \mathcal{K}_\varphi \simeq \mathbb{Q}_l[G]^m$ with $m = \mathrm{rk}_{\mathbb{Q}_l[G]}(\mathbb{Q}_l \otimes_{\mathbb{Z}} F^0) - \mathrm{rk}_{\mathbb{Q}_l[G]}(\mathbb{Q}_l \otimes_{\mathbb{Z}} F^1)$ (which, in turn, follows from diagram (4) or (5) below). Finally, if p is a rational prime such that $p \nmid \#G$, then $\mathbb{Z}_p[G]$ is regular and each finitely generated $\mathbb{Z}_p$-free $\mathbb{Z}_p[G]$-module is actually projective.

For the prime divisors p of $\#G$ (or for $p = l$ if $\Lambda = \mathbb{Z}_l$) we therefore apply the algorithm of [5, Sec. 4.2]. This algorithm either detects that $\mathcal{K}_\varphi$ is not locally free at p or computes a $\mathbb{Z}_p[G]$-basis. Alternatively we combine the algorithms of D.Holt which are already implemented in MAGMA [24] with [27, IX, Th. 8] to decide whether $\mathcal{K}_\varphi$ is c.t.. However, for our purposes the first method is preferable since we anyway need the local basis for the computation of the refined Euler characteristics.

Let $\mathbb{E}$ denote either $\mathbb{R}$ or $\mathbb{C}_l$. Let $\varepsilon(\varphi)$ be the perfect 2-extension represented by the bottom row of diagram (4) and set $\tau(\varphi) := (\varepsilon(\varphi), \lambda, \mathcal{L}^*)$. Applying the recipe for computing the refined Euler characteristic of trivialized perfect 2-extensions described above we obtain

$$\chi_{\Lambda[G],\mathbb{E}}(\tau(\varphi)) = [F^0, \lambda(\varphi)_{triv}, F^1 \oplus \mathcal{K}_\varphi] - \delta_\Lambda(\mathcal{L}^*) \in K_0(\Lambda[G], \mathbb{E})$$

where $\lambda(\varphi)_{triv}$ is the composite

$$\begin{aligned} F^0_{\mathbb{E}} &\xrightarrow{\sigma_1} Z_{\mathbb{E}} \oplus W_{\mathbb{E}} \xrightarrow{\sigma_2} \mathcal{K}_{\varphi,\mathbb{E}} \oplus H^0_{\mathbb{E}} \oplus W_{\mathbb{E}} \\ &\xrightarrow{\sigma_3} \mathcal{K}_{\varphi,\mathbb{E}} \oplus H^1_{\mathbb{E}} \oplus W_{\mathbb{E}} \xrightarrow{\sigma_4} \mathcal{K}_{\varphi,\mathbb{E}} \oplus F^1_{\mathbb{E}}. \end{aligned} \tag{5}$$

Here $\sigma_1, \sigma_2, \sigma_4$ are induced by choosing splittings of

$$\begin{aligned} 0 &\longrightarrow Z_{\mathbb{E}} \xrightarrow{\iota} F^0_{\mathbb{E}} \longrightarrow W_{\mathbb{E}} \longrightarrow 0, \\ 0 &\longrightarrow \mathcal{K}_{\varphi,\mathbb{E}} \longrightarrow Z_{\mathbb{E}} \xrightarrow{\varphi} H^0_{\mathbb{E}} \longrightarrow 0, \\ 0 &\longrightarrow W_{\mathbb{E}} \longrightarrow F^1_{\mathbb{E}} \xrightarrow{\pi} H^1_{\mathbb{E}} \longrightarrow 0, \end{aligned}$$

respectively, and finally σ_3 is induced by λ.

4. ETNC for l-power extensions

Let E be an elliptic curve defined over $\mathbb{Q}$ and K a finite Galois extension with group G. We always assume that the Tate-Shafarevic group $Ш(E/K)$ of E over K is finite. For a number field F we write G_F for the absolute Galois group. We let $T_l(E)$ denote the l-adic Tate module of E. We set $T_l := \mathbb{Z}_l[G] \otimes_{\mathbb{Z}_l} T_l(E)$ and regard it as a (left) module over $G_{\mathbb{Q}} \times G$, where $G_{\mathbb{Q}}$ acts diagonally and $g(\lambda \otimes t) = \lambda g^{-1} \otimes t$ for $g \in G, \lambda \in \mathbb{Z}_l[G]$ and $t \in T_l(E)$. Furthermore we define

$$V_l(E) := T_l(E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \quad V_l := T_l \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Let $S_{ram}(K/\mathbb{Q})$ be the set of primes which ramify in $K/\mathbb{Q}$ and $S_{bad}(E)$ the set of primes where E has bad reduction. We put $S := S_{ram}(K/\mathbb{Q}) \cup S_{bad}(E)$ and for a fixed prime l we set $S_l := S \cup \{l\}$.

We write $L(M_K, s)$ for the equivariant motivic L-function associated to $M_K = h^1(E_K)(1)$ and let $L^*(M_K)$ denote the leading coefficient in its Taylor expansion at $s = 0$ (see [11, Rem. 7 and Conj. 4]). For each absolutely irreducible character $\psi \in \mathrm{Irr}(G)$ we write $L(E/\mathbb{Q}, \psi, s)$ for the twisted Hasse-Weil-L-function. Assuming that $L(E/\mathbb{Q}, \psi, s)$ has analytic continuation to the complex plane we let $L^*(E/\mathbb{Q}, \psi, 1)$ denote the leading coefficient in its Taylor expansion at $s = 1$. We set $\mathcal{L}^* := \big(L^*(E/\mathbb{Q}, \bar{\psi}, 1)\big)_{\psi \in \mathrm{Irr}(G)}$ and recall from the paragraph preceding Remark 3.2 of [1] that $\mathcal{L}^* = L^*(M_K)$.

We write $I_p \leq G_{\mathbb{Q}}$ for the inertia subgroup at p and consider the complex

$$(T_l(E) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[G]^*)^{I_p} \xrightarrow{1 - Fr_p^{-1}} (T_l(E) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[G]^*)^{I_p} \tag{6}$$

with the non-trivial modules placed in degrees 0 and 1. If $(T_l(E) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[G]^*)^{I_p}$ is $\mathbb{Z}_l[G]$-perfect, then by [1, Rem. 3.2] the refined Euler characteristic associated with the above complex is given by $(L_p(E, \bar{\chi}, 1))_{\chi \in \mathrm{Irr}(G)}$.

If A is a semisimple algebra over a field, we write Nrd_A for the reduced norm map (as introduced, for example, in [11, Sec. 2.6]). Then we have

$$\mathrm{Nrd}_{\mathbb{C}[G]}\left(1 - Fr_p^{-1} \mid (V_l(E) \otimes_{\mathbb{Q}_l} \mathbb{Q}_l[G]^*)^{I_p}\right) = (L_p(E, \bar{\chi}, 1))_{\chi \in \mathrm{Irr}(G)} \in \zeta(\mathbb{C}[G])^\times.$$

As in the paragraph preceding Conjecture 3.5 in [1] we write $\mathrm{Reg} = \big(\mathrm{Reg}_\psi\big)_{\psi \in \mathrm{Irr}(G)}$ for the equivariant regulator, $R = (R_\psi)_{\psi \in \mathrm{Irr}(G)}$ for the vector of resolvents and $\Omega = (\Omega_\psi)_{\psi \in \mathrm{Irr}(G)}$ for the vector of periods. In this notation the equivariant periods are given by $R^{-1}\Omega$ (see [1, Prop. 3.1]). Note that the resolvent R depends on the choice of a normal basis element α_0 for $K/\mathbb{Q}$, so that R is only well defined modulo $\zeta(\mathbb{Q}[G])^\times$. The equivariant regulator Reg is defined as in [1, Rem. 2.6(b)]

with $Y^{ev} = E(K)_{\mathbb{Q}}$, $Y^{od} = E(K)^*_{\mathbb{Q}}$ and $\theta_{\mathbb{R}} = \lambda_{NT}$. We give here an explicit construction which is very much in the spirit of the present paper. To that end let $\mathbb{Q}[G]^{n_0} \xrightarrow{f} \mathbb{Q}[G]^{n_1} \longrightarrow E(K)^*_{\mathbb{Q}} \longrightarrow 0$ be a free $\mathbb{Q}[G]$-resolution of $E(K)^*_{\mathbb{Q}}$ and set $Z := \ker(f)$. Let $\varphi\colon Z \longrightarrow E(K)_{\mathbb{Q}}$ be any $\mathbb{Q}[G]$-homomorphism. By possibly increasing n_0 we may assume that φ is onto. We construct a diagram as in (4) and obtain an isomorphism $\lambda_{NT}(\varphi)_{triv}\colon \mathbb{R}[G]^{n_0} \longrightarrow \mathcal{K}_{\varphi,\mathbb{R}} \oplus \mathbb{R}[G]^{n_1}$ of $\mathbb{R}[G]$-modules as in (5). The $\mathbb{Q}[G]$-module $\mathcal{K}_\varphi$ is $\mathbb{Q}[G]$-free and we choose $\mathbb{Q}[G]$-basis of $\mathbb{Q}[G]^{n_0}$ and $\mathcal{K}_\varphi \oplus \mathbb{Q}[G]^{n_1}$. Representing $\lambda_{NT}(\varphi)_{triv}$ with respect to these basis we obtain a matrix $A_\varphi \in \mathrm{Gl}_{n_0}(\mathbb{R}[G])$ and we finally set

$$\mathrm{Reg} := \mathrm{Nrd}_{\mathbb{R}[G]}(A_\varphi) \cdot \zeta(\mathbb{Q}[G])^\times \in \zeta(\mathbb{R}[G])^\times/\zeta(\mathbb{Q}[G])^\times.$$

Adapting the arguments of [12] one can show that Reg is a well defined. Moreover, it is easily shown that this definition coincides with the one given in [1, Rem. 2.6(b)].

By abuse of notation we also write Reg for any lift to $\zeta(\mathbb{R}[G])^\times$ and set $u := \frac{\mathcal{L}^* R}{\Omega \mathrm{Reg}}$. Note that u is only well defined modulo $\zeta(\mathbb{Q}[G])^\times$. If we write $\mathrm{ETNC}_{\mathbb{Q}}$ for the rationality part of the ETNC, we have the explicit reformulation

$$\mathrm{ETNC}_{\mathbb{Q}} \text{ holds} \iff u \in \zeta(\mathbb{Q}[G])^\times. \tag{7}$$

Remark 4.1. We recall that $u = (u_\chi)_{\chi \in \mathrm{Irr}(G)}$ is a priori a vector of complex numbers. If $E(K)$ or a subgroup of finite index is explicitly known, then we can in principle compute the components of u to any required precision. If we also dispose of a good guess for bounds of the denominators, then we can use [1, Lem. 2.8] to check numerically whether $u \in \zeta(\mathbb{Q}[G])^\times$. We point out that in this way we cannot prove the rationality conjecture (even not for specific examples) but can only provide numerical evidence for it. However, in many examples, the complex numbers u_χ are very close to algebraic numbers and we round in a naive way (see Section 6 for an explicit example).

In Section 4 of [1] we developed methods to compute numerical evidence for the integrality part of the ETNC assuming the validity of the rationality conjecture and that $u \in \zeta(\mathbb{Q}[G])^\times$ is explicitly known. We showed that the precise knowledge of u can be used to prove the l-part of the integrality conjecture for almost all primes l.

We briefly recall the approach of loc. cit. The ETNC_l is formulated in terms of a perfect complex $R\Gamma_c(\mathbb{Z}_{S_l}, T_l)$ (see [11, Sec. 3.2-3.4]). To analyse this complex and to explicitly compute its cohomology one usually tries to define perfect complexes $R\Gamma_f(\mathbb{Q}, T_l)$ and $R\Gamma_f(\mathbb{Q}_p, T_l)$ for each $p \in S_l \cup \{\infty\}$ such that one has a true triangle

$$R\Gamma_c(\mathbb{Z}_{S_l}, T_l) \longrightarrow R\Gamma_f(\mathbb{Q}, T_l) \longrightarrow \bigoplus_{p \in S_l \cup \{\infty\}} R\Gamma_f(\mathbb{Q}_p, T_l).$$

This approach is motivated by work of Bloch and Kato and carried out in [10, Sec. 1.5.1] by Burns and Flach. However, if l divides $\#G$, it is not clear that it is always possible to define the complexes $R\Gamma_f(\mathbb{Q}, T_l)$ and $R\Gamma_f(\mathbb{Q}_p, T_l)$ so that they are perfect (see the comment at the beginning of Sec. 1.5.1 of [10]). For that reason one is forced to introduce additional hypothesis which do not occur in the general statement of the ETNC.

For a finite place v of K we write $\mathcal{O}_{K_v}$ for the valuation ring in the completion K_v and $\mathfrak{m}_v$ for the maximal ideal. Let $k_v := \mathcal{O}_{K_v}/\mathfrak{m}_v$ denote the residue class field. We write $E_0(K_v)$ for the points of $E(K_v)$ which reduce to a non-singular point on the reduced curve $\bar{E}$. Let $\bar{E}_{ns}(k_v)$ denote the group of non-singular points of $\bar{E}(k_v)$.

Let $\bar{I}_p \leq G$ denote the inertia subgroup at p. For the approach in [1] we used the following

Hypothesis:

(H0) $Ш(E/K)$ is finite.
(H1) l is at most tamely ramified in $K/\mathbb{Q}$.
(H2) (a) If $l \in S$ or $l = 2$, then $l \nmid \#G$.
(b) If $l \notin S$ and $l \neq 2$, then $l \nmid \bar{I}_p$ for all $p \in S$.
(H3) $S_{bad}(E) \cap S_{ram}(K/\mathbb{Q}) = \emptyset$.
(H4) If $l \mid \#G$, then
(a) $E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_l$, $(E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_l)^*$ are $\mathbb{Z}_l[G]$-perfect and $l \nmid \#E(K)_{tors}$.
(b) $l \nmid \#Ш(E/K)$.
(H5) If $l \notin S$ and $l \neq 2$, then $l \nmid \#(E(K_v)/E_0(K_v))$ for all $v \in S(K)$.

Note that only (H2) and (H3) were needed to show that the complexes $R\Gamma_f(\mathbb{Q}, T_l)$ and $R\Gamma_f(\mathbb{Q}_p, T_l)$ are perfect (see [1, Lemma 4.1]). Hypothesis (H1), (H4) and (H5) guaranteed that all of the cohomology groups of the complexes $R\Gamma_f(\mathbb{Q}, T_l)$ and $R\Gamma_f(\mathbb{Q}_p, T_l)$ were perfect. By [11, Prop. 2.1 (4)] we can therefore work entirely with the cohomology modules and avoid the numerical computation of the complexes. However, the above hypothesis force to exclude finitely many primes l from our considerations.

For prime divisors l of $\#G$ the condition (H4)(a) forced $\mathrm{rk}(E(K)) = 0$, or in other words, we had to exclude prime divisors l of $\#G$ whenever E/K has non-trivial Mordell-Weil rank. In this note we will focus on primes l dividing $\#G$ and also consider curves E with $\mathrm{rk}(E(K)) > 0$.

Following [11, (1.38)] we define

$$C(\mathbb{Q}_p, T_l(E)) := H^0\left(\mathbb{Q}_p, H^1\left(I_p, T_l(E)\right)_{tors}\right)$$

and set $c_p(T_l(E)) := \#C(\mathbb{Q}_p, T_l(E))$. We write $c_p(E)$ for the usual Tamagawa numbers and note that by [19, Exp. IX, (11.3.8)] for $p \neq l$ the number $c_p(T_l(E))$ is the l-primary part of $c_p(E)$.

We will combine the approach of [1] with Proposition 4.3.1 of [13]. To that end we replace our hypothesis (H1)-(H5) by hypothesis (i)-(vi) from the introduction and recall that (i)-(v) essentially coincide with condition (B) of [13]. Note that our (v) slightly differs from Burns' hypothesis (v). However, our condition (v) implies that the modules $C(\mathbb{Q}_p, T_l(E))$ are trivial for all $p \mid N_E$ and this is exactly the assumption we need for the proof of [13, Lemma 12.2.2].

For the definition of the complexes $R\Gamma_f(\mathbb{Q}_p, T_l)$ we refer the reader to [13, Sec. 12]. Note that by [13, Rem. 12.4.2] this definition essentially coincides with the definitions of [1]. We can still compute elements u_l and ξ_l as in [1, Prop. 4.4], however, there are some changes in the computation which we indicate in the following. But recall first of all that by [1, Prop. 4.4] we have

$$\mathrm{ETNC}_l \text{ holds } \iff u_l \equiv \xi_l \left(\mathrm{mod}\ \mathrm{Nrd}_{\mathbb{Q}_l[G]}(\mathbb{Z}_l[G]^\times)\right). \tag{8}$$

The changes concern the computation of the Euler characteristics

a) $\chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(R\Gamma_f(\mathbb{Q}, T_l), \lambda_{NT}^{-1})$,
b) $\chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(R\Gamma_f(\mathbb{Q}_p, T_l), 0)$ for $p \neq l, \infty$,

Note that the λ_{NT} was denoted by δ in [1].

We first look at b). Hypothesis (ii) and (iv) imply $l \notin S$. If $p \notin S_{ram}(K/\mathbb{Q})$, then the definitions of $R\Gamma_f(\mathbb{Q}_p, T_l)$ in [13, Sec. 12.2] and [1, Sec. 4] coincide. Explicitly, $R\Gamma_f(\mathbb{Q}_p, T_l)$ is given by the complex (6) and the refined Euler characteristic is $(L_p(E/\mathbb{Q}, \bar{\chi}, 1))_{\chi \in \mathrm{Irr}(G)}$. Assuming (H2)(b) the same would be true for all $p \in S$.

On the other hand, if $p \in S_{ram}(K/\mathbb{Q})$, then l violates (H2)(b) and the computation in [1] is no longer valid. In contrast, by [13, Lemma 12.2.1] $R\Gamma_f(\mathbb{Q}_p, T_l)$ as defined in [13] is perfect, and in fact, the proof shows that as a consequence of (iii) $R\Gamma_f(\mathbb{Q}_p, T_l)$ is actually acyclic. Therefore the refined Euler characteristic $\chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(R\Gamma_f(\mathbb{Q}_p, T_l), 0)$ is trivial in this case.

Recall the definition of ξ_l in [1, Prop. 4.4] and its proof. Assuming (H2)(b) the Euler factors arising from $R\Gamma_f(\mathbb{Q}_p, T_l)$ cancelled with Euler factors arising from certain identifications made in [11] for all $p \in S$.

Assuming our hypothesis (i)-(v) instead of (H0)-(H5) the Euler factors for $p \in S_{ram}(K/\mathbb{Q})$ survive and we have to set

$$\xi_l := \prod_{p \in S_{ram}(K/\mathbb{Q})} (L_p(E, \bar{\chi}, 1))^{-1}_{\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)}$$

where $L_p(E, \chi, 1)$ denotes the local Euler factor at $s = 1$. Indeed, since by assumption $l \nmid \#E(K)_{tors}$ and $l \nmid \#Ш(E/K)$ the ξ_l of [1, Prop. 4.4] is trivial and we just obtain the Euler factors as explained above. Note that we did not use (v) for this computation.

We now turn to the computation of the Euler characteristic in a). We point out that condition (v) is needed to show that $R\Gamma_f(\mathbb{Q}, T_l)$ is perfect (see the proof of [13, Lemma 12.2.2]). In principle we could work in the generality of [13], but for simplicity we introduce the additional hypothesis (vii) and (viii) from the introduction. These conditions will substantially simplify the computation of the l-integral equivariant regulator.

Lemma 4.2. *Assume (i) - (viii). Then* $E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_l = E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$.

Proof. Since G is a l-group condition (iii) implies that $l \nmid \#E(K)_{tors}$. Therefore $E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_l$ and $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$ are both torsion free. By the elementary divisor theorem we may find $\mathbb{Z}_l$-basis $P_1, \ldots, P_r$ of $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$ and $Q_1, \ldots, Q_r$ of $E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_l$ such that

$$P_i = l^{n_i} Q_i \text{ with } 0 \leq n_1 \leq n_2 \leq \ldots \leq n_r.$$

For each $\sigma \in G$ and each i one then has $l^{n_i}(Q_i^\sigma - Q_i) = 0$, and therefore $Q_i^\sigma = Q_i$. This implies $Q_i \in E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$. □

The following should be considered as a variant of [13, Prop. 4.3.1]. The Euler characteristic $\chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(R\Gamma_f(\mathbb{Q}, T_l), \lambda_{NT}^{-1})$ was computed in [1, Lemma 4.2] under the assumption (H0) - (H5). As in [1, Sec. 4.2] we see that by our new assumptions (i) - (viii) the only non trivial cohomolgy groups of $R\Gamma_f(\mathbb{Q}, T_l)$ are given by

$$H^1_f(\mathbb{Q}, T_l) \simeq E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_l, \quad H^2_f(\mathbb{Q}, T_l) \simeq (E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_l)^*.$$

Since $H^1_f(\mathbb{Q}, T_l)$ and $H^2_f(\mathbb{Q}, T_l)$ are no longer assumed to be perfect, the computation of $\chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(R\Gamma_f(\mathbb{Q}, T_l), \lambda_{NT}^{-1})$ has to change.

We set $H^0 := E(K)$, $H^1 := E(K)^*$, $H^0_l := H^0 \otimes_{\mathbb{Z}} \mathbb{Z}_l$ and $H^1_l := H^1 \otimes_{\mathbb{Z}} \mathbb{Z}_l$. We recall from [13, Sec. 3.1] that the shifted complex $C^\bullet := R\Gamma_f(\mathbb{Q}, T_l)[1]$ can be

represented by a perfect 2-extension ε_{τ_0} of H^1_l by H^0_l. In a little more detail, the tautological exact sequence

$$0 \longrightarrow H^0_l \longrightarrow C^0/B^0(C^\bullet) \longrightarrow Z^1(C^\bullet) \longrightarrow H^1_l \longrightarrow 0$$

where $B^i(C^\bullet)$ and $Z^i(C^\bullet)$ are the boundaries and cocycles of $C^\bullet$ specifies an element $\varepsilon_{\tau_0} \in \mathrm{Ext}^2_{\mathbb{Z}_l[G]}(H^1_l, H^0_l)$. Using [25, Lemma 8.17] it is straightforward to check that ε_{τ_0} is perfect. But because of the non explicit definition of the complex $R\Gamma_f(\mathbb{Q}, T_l)$ (see [13, (53), (54)] or [10, (1.33)]) it seems to be very difficult to describe the extension class ε_{τ_0} more precisely.

Consider now the augmented trivialized extension $\tau_0 = (\varepsilon_{\tau_0}, \lambda_{NT}, 1)$. Then

$$\chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(R\Gamma_f(\mathbb{Q}, T_l), \lambda_{NT}^{-1}) = \chi_{\mathbb{Z}_l[G],\mathbb{C}_l}(\tau_0).$$

Together with (8) this proves Theorem 1.1.

We first consider the case $\mathrm{rk}(E(K)) = 0$. Then (viii) is automatically fulfilled and the Ext-group is obviously trivial. We fix a $\mathbb{Z}_l[G]$-basis α_0 of the localization $\mathcal{O}_{K,l} = \mathbb{Z}_l \otimes_{\mathbb{Z}} \mathcal{O}_K$ (which exists since l is unramified in $K/\mathbb{Q}$) and let $R = R(\alpha_0)$ denote the equivariant resolvent with respect to α_0. We set

$$u_l := \frac{\mathcal{L}^* R}{\Omega} \tag{9}$$

and note that u_l is well defined modulo $\mathrm{Nrd}_{\mathbb{Q}_l[G]}(\mathbb{Z}_l[G]^\times)$. Recall that we assume (7) and that we can compute the exact values of $u_l \in \zeta(\mathbb{Q}[G])^\times$. Note also that we do not use (vii) in this case, so that this proves Corollary 1.2.

If $\mathrm{rk}(E(K)) > 0$ we are not able to pin down the extension class ε_{τ_0} more concretely and therefore will compute the refined Euler characteristic for all augmented trivialized 2-extensions $\tau = (\varepsilon, \lambda_{NT}, 1)$ with perfect $\varepsilon \in \mathrm{Ext}^2_{\mathbb{Z}_l[G]}(H^1_l, H^0_l)$.

By the methods introduced in Section 2 we could do this without assuming (vii) and (viii), however, with these assumptions we can do a little better. We fix a $\mathbb{Z}$-basis $P_1, \ldots, P_r$ of $E(\mathbb{Q})_{tf}$ and write $P_1^*, \ldots, P_r^*$ for the dual basis (i.e. $P_i^*(P_j) = \delta_{ij}$). By Lemma 4.2 we may identify the modules H^0_l and H^1_l with $\mathbb{Z}_l^r$ by this choice of basis.

We set $N_G := \sum_{g \in G} g$ and consider the standard exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_l & \xrightarrow{N_G} & \mathbb{Z}_l[G] & \xrightarrow{g_0 - 1} & \mathbb{Z}_l[G] & \xrightarrow{aug} & \mathbb{Z}_l \longrightarrow 0 \\ & & & & & \searrow \quad \nearrow & & & \\ & & & & & W & & & \end{array} \tag{10}$$

with $W = \mathbb{Z}_l[G](g_0 - 1)$. Let $e_0 = \frac{1}{\#G} N_G$ and $e_1 = 1 - e_0$. Then $W_{\mathbb{Q}_l} = \mathbb{Q}_l[G]e_1$ and we have a splitting of $\mathbb{Q}_l[G] \longrightarrow W_{\mathbb{Q}_l} = \mathbb{Q}_l[G]e_1$ defined by $e_1 \mapsto \frac{1}{g_0 - 1} e_1$. This splitting induces the isomorphism

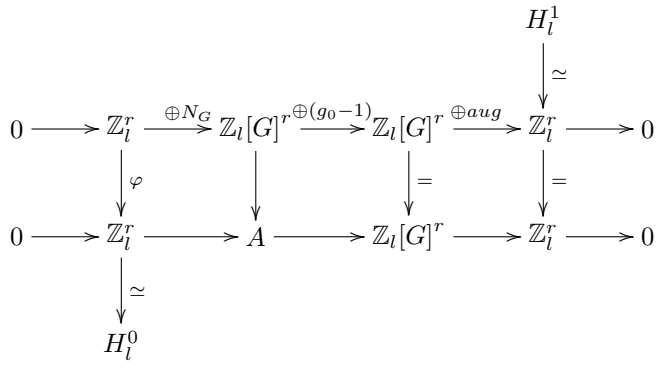
$$\mathbb{Q}_l[G] \simeq \mathbb{Q}_l \oplus W_{\mathbb{Q}_l}, \quad 1 \mapsto (\frac{1}{\#G}, g_0 - 1). \tag{11}$$

For the splitting of $\mathbb{Q}_l[G] \xrightarrow{aug} \mathbb{Q}_l$ we use $1 \mapsto e_0$. Then

$$\mathbb{Q}_l[G] \simeq \mathbb{Q}_l \oplus W_{\mathbb{Q}_l}, \quad 1 \mapsto (1, e_1). \tag{12}$$

We take r copies of the sequence (10) and use this to compute $\mathrm{Ext}^2_{\mathbb{Z}_l[G]}(H^1_l, H^0_l)$. For each map $\varphi \in \mathrm{Hom}_{\mathbb{Z}_l[G]}(\mathbb{Z}_l^r, \mathbb{Z}_l^r)$ we obtain a commutative diagram of the form

(13)



Lemma 4.3. *a) Each element of* $\mathrm{Ext}^2_{\mathbb{Z}_l[G]}(\mathbb{Z}_l^r, \mathbb{Z}_l^r)$ *can be represented by an injective map* φ.

b) Suppose that φ *is injective. Then* A *is c.t. if and only if* φ *is surjective (or equivalenty, if and only if* $l \nmid \det_{\mathbb{Z}_l}(\varphi)$*).*

Proof. a) We recall (for example from [9, Ch. III, Prop. (2.2)]) that for a $\mathbb{Z}_l$-free $\mathbb{Z}_l[G]$-module X and a $\mathbb{Z}_l[G]$-module Y one has $H^i(G, \mathrm{Hom}_{\mathbb{Z}_l}(X,Y)) \simeq \mathrm{Ext}^i_{\mathbb{Z}_l[G]}(X,Y)$ for all $i \geq 1$. It follows that $\mathrm{Ext}^i_{\mathbb{Z}_l[G]}(X,Y)$ is annihilated by $\#G$. Therefore φ and $\varphi + Nid$ define the same element in the Ext-group for all $N \in \mathbb{N}$ which are divisible by $\#G$. If $-N$ is not an eigenvalue of φ, then $\varphi + Nid$ is injective.

b) If φ is an isomorphism, then $A \simeq \mathbb{Z}_l[G]^r$. Conversely, if A is c.t., then the finite module $\mathrm{cok}(\varphi)$ is also c.t. Now we use the fact that a finite l-group C with trivial G-action is c.t. if and only if $C = 0$. □

Following the recipe in Section 3, see in particular (5), we have to compute the element $[\mathbb{Z}_l[G]^r, \lambda_{NT}(\varphi)_{triv}, \mathbb{Z}_l[G]^r] \in K_0(\mathbb{Z}_l[G], \mathbb{C}_l)$, where $\lambda_{NT}(\varphi)_{triv}$ is the following composite of isomorphisms

$$\mathbb{C}_l[G]^r \xrightarrow{(11)} (\mathbb{C}_l)^r \oplus W^r_{\mathbb{C}_l} \xrightarrow{(\varphi, id)} E(K)_{\mathbb{C}_l} \oplus W^r_{\mathbb{C}_l} \xrightarrow{(\lambda_{NT}, id)} E(K)^*_{\mathbb{C}_l} \oplus W^r_{\mathbb{C}_l} \xrightarrow{(12)} \mathbb{C}_l[G]^r.$$

Let $e_1, \ldots, e_r$ denote the standard basis of $\mathbb{Z}_l^r$ and write $\Phi \in \mathrm{Gl}_r(\mathbb{Z}_l)$ for the coordinate matrix of φ, explicitly $\varphi(e_i) = \sum_{j=1}^r \Phi_{ji} e_j$. We also recall the definition of $\lambda_{NT} : H^0_l \longrightarrow H^1_l$. Explicitly, $\lambda_{NT}(P) = \langle P, \cdot \rangle$ for $P \in E(K)$, where $\langle \cdot, \cdot \rangle$ denotes the height pairing. Then one has

$$\lambda_{NT}(P_j) = \sum_{k=1}^r \langle P_k, P_j \rangle P_k^*, \quad j = 1, \ldots, r.$$

We write $w_1, \ldots, w_r$ for the standard basis of $\mathbb{C}_l[G]^r$ and set

$$\Psi := \left(\langle P_k, P_j \rangle \right)_{1 \leq k,j \leq r}.$$

Then

$$\begin{aligned}
w_i &\overset{(11)}{\mapsto} \left((0,\dots,0,\frac{1}{\#G},0,\dots,0),(0,\dots,0,g_0-1,0,\dots,0)\right)\\
&\overset{(\varphi,id)}{\mapsto} \left(\frac{1}{\#G}\sum_{j=1}^{r}\Phi_{ji}P_j,(0,\dots,0,g_0-1,0,\dots,0)\right)\\
&\overset{(\lambda_{NT},id)}{\mapsto} \left(\frac{1}{\#G}\sum_{j=1}^{r}\Phi_{ji}\sum_{k=1}^{r}\langle P_k,P_j\rangle P_k^*,(0,\dots,0,g_0-1,0,\dots,0)\right)\\
&\overset{(12)}{\mapsto} \left(\frac{1}{\#G}\sum_{j=1}^{r}\Phi_{ji}\langle P_k,P_j\rangle e_0\right)_{k=1,\dots,r}+(0,\dots,0,g_0-1,0,\dots,0)\\
&= \left(\frac{1}{\#G}\left(\Psi\Phi\right)_{ki}e_0\right)_{k=1,\dots,r}+(0,\dots,0,g_0-1,0,\dots,0)\\
&= \left((g_0-1)+\frac{1}{\#G}\left(\Psi\Phi\right)_{ii}e_0\right)w_i+\sum_{k=1,k\neq i}^{r}\left(\frac{1}{\#G}\left(\Psi\Phi\right)_{ki}e_0\right)w_k.
\end{aligned}$$

With respect to the basis $w_1,\dots,w_r$ of $\mathbb{C}_l[G]^r$ the map $\lambda_{NT}(\varphi)_{triv}$ is therefore represented by the matrix

$$\begin{pmatrix} g_0-1 & & \\ & \ddots & \\ & & g_0-1 \end{pmatrix}+\left(\frac{1}{\#G}\left(\Psi\Phi\right)_{ik}e_0\right)_{1\leq i,k\leq r}$$

Upon taking determinants (which in this commuataive case is the same as taking reduced norms) one obtains

$$\det_{\mathbb{Z}_l[G]}(\lambda_{NT}(\varphi)_{triv})=(g_0-1)^r+\frac{1}{(\#G)^r}\det(\Psi)\det(\Phi)e_0.$$

Hence we get for the l-integral equivariant regulator $\mathrm{Reg}(\varphi)=\left(\mathrm{Reg}_\chi(\varphi)\right)_{\chi\in\mathrm{Irr}(G)}$

$$\mathrm{Reg}_\chi(\varphi)=\begin{cases}(\chi(g_0)-1)^r, & \text{if } \chi \text{ is non-trivial,}\\ \frac{1}{(\#G)^r}\det(\Psi)\det(\Phi), & \text{if } \chi \text{ is trivial.}\end{cases}$$

We set

$$u_l(\varphi):=\frac{\mathcal{L}^*R}{\Omega\mathrm{Reg}(\varphi)}=\frac{\mathcal{L}^*R}{\Omega\mathrm{Reg}(id)}\cdot\frac{1}{E(\varphi)}$$

with

$$E(\varphi)=\left(E_\chi(\varphi)\right),\quad E_\chi(\varphi)=\begin{cases}1, & \text{if } \chi \text{ is non-trivial,}\\ \det(\Phi), & \text{if } \chi \text{ is trivial,}\end{cases}\tag{14}$$

and recall once again that $R=R(\alpha_0)$ must be computed with respect to a $\mathbb{Z}_l[G]$-basis of $\mathcal{O}_{K,l}$.

From diagram (13) it is clear that $\mathrm{Ext}^2_{\mathbb{Z}_l[G]}(H^1_l,H^0_l)\simeq M_r(\mathbb{Z}_l/\#G\mathbb{Z}_l)$, the ring of $r\times r$ matrices. We conclude from Lemma 4.3 that the perfect extension classes are represented by the elements of $\mathrm{Gl}_r(\mathbb{Z}_l/\#G\mathbb{Z}_l)$. We write

$$\delta_l\colon \mathbb{Q}_l[G]^\times/\mathrm{Nrd}_{\mathbb{Q}_l[G]}(K_1(\mathbb{Z}_l[G]))\longrightarrow K_0(\mathbb{Z}_l[G],\mathbb{Q}_l)$$

for the canonical isomorphism (see e.g. [5, Th. 2.3 (ii)] where its inverse is denoted by $\bar{n}_{\mathcal{A}}$). We also recall that the canonical map $\mathbb{Z}_l[G]^\times \longrightarrow K_1(\mathbb{Z}_l[G])$ is surjective because $\mathbb{Z}_l[G]$ is a semilocal ring.

Recall that $\varepsilon_{\tau_0} \in \mathrm{Ext}^2_{\mathbb{Z}_l[G]}(H^1_l, H^0_l)$ represents the extension class of the perfect complex $C^\bullet = R\Gamma_f(\mathbb{Q}, T_l)[1]$. Let $\varphi_0 : \mathbb{Z}_l^r \longrightarrow \mathbb{Z}_l^r$ be the map which represents ε_{τ_0} when $\mathrm{Ext}^2_{\mathbb{Z}_l[G]}(H^1_l, H^0_l)$ is computed using the top exact sequence of diagram (13).

As a consequence of (8) we obtain

$$\mathrm{ETNC}_l \text{ holds} \iff \delta_l(u_l(\varphi_0)\xi_l^{-1}) = 0. \tag{15}$$

Definition 4.4. We set

$$\mathcal{E} := \{\delta_l\left(E(\varphi)\right) \mid \Phi \in \mathrm{Gl}_r(\mathbb{Z}_l/\#G\mathbb{Z}_l)\}.$$

From (14) together with [5, Th. 2.3 (i), Th. 2.4 (iv)] we deduce that $\mathcal{E}$ is a subgroup of $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors}$ of order $l^{n-1}(l-1)$.

We summarize the preceeding discussion in the next theorem which includes the assertions of Theorem 1.3.

Theorem 4.5. *We assume (i) - (viii). Then the following are equivalent:*

(i) *The ETNC$_l$ is true modulo $\mathcal{E}$.*
(ii) $\mathcal{E} = \left\{\delta_l\left(u_l(\varphi)\xi_l^{-1}\right) \mid \Phi \in \mathrm{Gl}_r(\mathbb{Z}_l/\#G\mathbb{Z}_l)\right\}$.
(iii) $\delta_l\left(u_l(id)\xi_l^{-1}\right) \in \mathcal{E}$.

Proof. We recall that $\delta_l\left(u_l(\varphi)\right) = \delta_l\left(u_l(id)\cdot\frac{1}{E(\varphi)}\right)$.

We first proof that (i) implies (ii). If ETNC$_l$ is valid modulo $\mathcal{E}$ then it follows from (15) and the definition of $\mathcal{E}$ that there is an isomorphism $\psi : \mathbb{Z}_l^r \longrightarrow \mathbb{Z}_l^r$ such that $\delta_l\left(u_l(\varphi_0)\xi_l^{-1}\right) = \delta_l\left(E(\psi)\right)$. Then

$$\begin{aligned}
& \{\delta_l\left(u_l(\varphi)\xi_l^{-1}\right) \mid \Phi \in \mathrm{Gl}_r(\mathbb{Z}_l/\#G\mathbb{Z}_l)\} \\
= \; & \{\delta_l\left(u_l(id)\xi_l^{-1}E(\varphi)^{-1}\right) \mid \Phi \in \mathrm{Gl}_r(\mathbb{Z}_l/\#G\mathbb{Z}_l)\} \\
= \; & \{\delta_l\left(u_l(\varphi_0)\xi_l^{-1}E(\varphi)^{-1}\right) \mid \Phi \in \mathrm{Gl}_r(\mathbb{Z}_l/\#G\mathbb{Z}_l)\} \\
= \; & \{\delta_l\left(E(\psi)E(\varphi)^{-1}\right) \mid \Phi \in \mathrm{Gl}_r(\mathbb{Z}_l/\#G\mathbb{Z}_l)\} \\
= \; & \mathcal{E}.
\end{aligned}$$

(ii) obviously implies (iii). Finally suppose that (iii) holds. Then there exists $\psi : \mathbb{Z}_l^r \longrightarrow \mathbb{Z}_l^r$ such that $\delta_l\left(u_l(id)\xi_l^{-1}\right) = \delta_l\left(E(\psi)\right)$. Hence

$$\delta_l\left(u_l(\varphi_0)\xi_l^{-1}\right) = \delta_l\left(\frac{E(\psi)}{E(\varphi_0)}\right) \in \mathcal{E}.$$

□

By the methods of [5] we can numerically test (ii) or (iii) of Theorem 4.5 and thus verify ETNC$_l$ modulo the subgroup $\mathcal{E}$.

If $r = 0$ the subgroup $\mathcal{E}$ is trivial and we can numerically fully verify the validity of the ETNC$_l$. In the next section we will express the validity of the ETNC$_l$ in terms of explicit congruences.

For $r > 0$ we are not able to fully verify the ETNC. We note that for $n = 1$ the cardinality of $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors}$ is equal to $l - 1$ by [5, Cor. 8.2]. Therefore $\mathcal{E} = K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors}$, so that in this case we study the ETNC$_l$ modulo torsion.

However, for $n \geq 2$ our computational results have a much stronger meaning because, as we will prove in the next section,

$$\#K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors} = (l-1)^n l^{\frac{l^n-1}{l-1}-n} \gg l^{n-1}(l-1) = \#\mathcal{E}.$$

We end this section by very explicitly describing what has to be done to verify the ETNC$_l$ modulo $\mathcal{E}$. Let $G = \langle g_0 \rangle$. We fix a primitive l^n-th root of unity ζ_{l^n} and for $m = 0, \ldots, n$ we set $\zeta_{l^m} := \zeta_{l^n}^{l^{n-m}}$. For $i = 0, \ldots, n$ we define an irreducible character χ_i by $\chi_i(g_0) := \zeta_{l^i}$. Then

$$\mathbb{Q}_l[G] \simeq \bigoplus_{i=0}^{n} \mathbb{Q}_l(\zeta_{l^i}), \quad \lambda \mapsto (\chi_i(\lambda))_i \,. \tag{16}$$

and via this identification the maximal order $\mathcal{M}$ of $\mathbb{Q}_l[G]$ is identified with $\bigoplus_{i=0}^{n} \mathbb{Z}_l[\zeta_{l^i}]$. We will often consider this identification as an equality.

For a positive integer k with $l \nmid k$ we write σ_k for the Galois automorphism which sends ζ_{l^i} to $\zeta_{l^i}^k$. Then the rationality conjecture (7) holds, if and only if for each $i \in \{0, \ldots, n\}$ and each $k = 1, \ldots, l^{i-1}(l-1)$ with $l \nmid k$ one has

$$\frac{L^*(E/\mathbb{Q}, \bar{\chi}_i^k, 1) R_{\chi_i^k}}{\Omega_{\chi_i^k} \mathrm{Reg}_{\chi_i^k}(id)} \in \mathbb{Q}(\zeta_l^i) \tag{17}$$

and

$$\frac{L^*(E/\mathbb{Q}, \bar{\chi}_i^k, 1) R_{\chi_i^k}}{\Omega_{\chi_i^k} \mathrm{Reg}_{\chi_i^k}(id)} = \left(\frac{L^*(E/\mathbb{Q}, \bar{\chi}_i, 1) R_{\chi_i}}{\Omega_{\chi_i} \mathrm{Reg}_{\chi_i}(id)} \right)^{\sigma_k}. \tag{18}$$

We point out once again that we can only provide numerical evidence for the rationality conjecture. Nevertheless, if we compute good complex approximations and if we have a guess for the denominator, then we can compute $\frac{L^*(E/\mathbb{Q}, \bar{\chi}_i, 1) R_{\chi_i}}{\Omega_{\chi_i} \mathrm{Reg}_{\chi_i}(id)}$ as an element of $\mathbb{Q}(\zeta_{l^i})$. Note also that for the computation of $R_{\chi_i} = R_{\chi_i}(\alpha_0)$ we need an l-integral normal basis α_0 element of $\mathcal{O}_K$. Heuristically one obtains the best results (in the naive meaning that we get small algebraic numbers with small denominators) if we use an integral normal basis element α_0. Such an element can be computed by the algorithms developed in [3] and [4].

Henceforth we assume the rationality conjecture and set

$$\eta_i := \frac{L^*(E/\mathbb{Q}, \bar{\chi}_i, 1) R_{\chi_i}}{\Omega_{\chi_i} \mathrm{Reg}_{\chi_i}(id)} \cdot \prod_{p \in S_{ram}(K/\mathbb{Q})} L_p(E/\mathbb{Q}, \bar{\chi}_i, 1). \tag{19}$$

The vector $\eta := (\eta_0, \ldots, \eta_n)$ represents $u_l(id)\xi_l^{-1}$ via the Wedderburn decomposition (16).

Now the l-part of the ETNC holds modulo the torsion subgroup $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors}$, if and only if $\eta_i \in \mathbb{Z}_l[\zeta_{l^i}]^\times$ for $i = 0, \ldots, n$. Finally, the l-part of the ETNC is valid modulo $\mathcal{E}$, if and only if $\eta \in \mathcal{E}$. In the case $r = 0$ we can be more precise, because $\mathcal{E}$ is trivial. In this case we obtain that the ETNC$_l$ is true if and only $(\eta_0, \ldots, \eta_n)$ satisfies the recursive congruences which we will describe in the next section. In the simplest case when $n = 1$ we obtain that the l-part is true, if and only if $\eta_1 \equiv \eta_0 (\mathrm{mod}\ (1 - \zeta_l))$.

Remark 4.6. In the case $r = 0$ one can use the theory of modular symbols to compute the precise value of $\frac{\tau(\chi) L(E/\mathbb{Q}, \chi, 1)}{\Omega_\chi}$ where $\tau(\chi)$ denotes a certain Gauss sum (see e.g. [16, Prop. 2.3]). Studying the relation between Gauss sums and the

resolvents used in [1] and in this manuscript it seems to be possible to provide proofs for ETNC_l by combining results on BSD for $E/\mathbb{Q}$ (e.g. from [28]) with our Theorem 4.5. This will be the subject of a further research project.

5. Relative K-groups for cyclic l-groups

Let l be a prime and G an arbitrary finite group. We let $\mathcal{M} \subseteq \mathbb{Q}_l[G]$ denote a maximal order which contains $\mathbb{Z}_l[G]$ and write $C = \zeta(\mathbb{Q}_l[G])$ for the center of $\mathbb{Q}_l[G]$. We write $\mathcal{O}_C$ for the integral closure of $\mathbb{Z}_l$ in C and recall that $\mathcal{O}_C = C \cap \mathcal{M}$. From [5, Th. 2.4] we obtain

$$K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors} \simeq \mathcal{O}_C^\times / \mathrm{Nrd}_{\mathbb{Q}_l[G]}(\mathbb{Z}_l[G]^\times).$$

Let now $G = \langle g_0 \rangle$ be cyclic of order l^n, $n \geq 1$. We fix a primitive l^n-th root of unity ζ_{l^n} and set $\zeta_{l^m} := \zeta_{l^n}^{l^{n-m}}$ for $m = 0, \ldots, n$. Consider the Wedderburn decomposition (16). Recall that we identify $\mathcal{M}$ and $\oplus_{i=0}^n \mathbb{Z}_l[\zeta_{l^i}]$. The basic question is now to decide which $(n+1)$-tuples $(\gamma_0, \ldots, \gamma_n) \in \mathcal{M}^\times$ are actually contained in $\mathbb{Z}_l[G]^\times$.

It is well known that for $n = 1$ one has $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors} \simeq \mathbb{F}_l^\times$ (see [5, Cor. 8.2]) and that $(\gamma_0, \gamma_1) \in \mathbb{Z}_l[G]$, if and only if $\gamma_1 \equiv \gamma_0 (\mathrm{mod}\ (1 - \zeta_l))$.

In this section we compute the order of $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors}$ for arbitrary n and develop a recursive test which describes the image of $\mathbb{Z}_l[G]$ in $\mathcal{O}_C$ in terms of explicit congruences.

Let $m \in \{1, \ldots, n\}$ and set $C_{l^m} := \langle g_0 \bmod \langle g_0^{l^m} \rangle \rangle$. The results of this section are based on the following cartesian square

$$\begin{array}{ccc} \mathbb{Z}_l[C_{l^m}] & \longrightarrow & \mathbb{Z}_l[C_{l^{m-1}}] \\ \downarrow & & \downarrow \\ \mathbb{Z}_l[\zeta_{l^m}] & \longrightarrow & \mathbb{F}_l[C_{l^{m-1}}]. \end{array} \tag{20}$$

This cartesian square is best understood in terms of polynomial rings. Let $\Phi_{l^m}(T)$ be the l^m-th cyclotomic polynomial. From

$$\Phi_{l^m}(T) \equiv l (\mathrm{mod}\ (T^{l^{m-1}} - 1))$$

we immediately deduce the following equality of $\mathbb{Z}_l[T]$-ideals

$$\left(\Phi_{l^m}(T), T^{l^{m-1}} - 1\right) = \left(l, T^{l^{m-1}} - 1\right). \tag{21}$$

Now consider the pull back of

$$\begin{array}{ccc} & & \mathbb{Z}_l[T]/(T^{l^{m-1}} - 1) \\ & & \Big\downarrow \bmod l \\ \mathbb{Z}_l[T]/(\Phi_{l^m}(T)) & \xrightarrow[\bmod\ (l, T^{l^{m-1}} - 1)]{} & \mathbb{F}_l[T]/(T^{l^{m-1}} - 1). \end{array}$$

We shall prove that this pull back is canonically isomorphic to $\mathbb{Z}_l[T]/(T^{l^m} - 1)$.

Lemma 5.1. *The canonical map*

$$\begin{array}{rcl} \dfrac{\mathbb{Z}_l[T]}{(T^{l^m}-1)} & \longrightarrow & \left\{(f,g) \in \dfrac{\mathbb{Z}_l[T]}{(T^{l^{m-1}}-1)} \oplus \dfrac{\mathbb{Z}_l[T]}{(\Phi_{l^m}(T))} \mid f \equiv g (\mathrm{mod}\ (l, T^{l^{m-1}}-1))\right\} \\ h & \mapsto & \Big(h \bmod (T^{l^{m-1}}-1), h \bmod (\Phi_{l^m}(T))\Big) \end{array}$$

is an isomorphism.

Proof. We first prove injectivity. Suppose that $h \in \left(T^{l^{m-1}}-1\right) \cap (\Phi_{l^m}(T))$. Let $a(T) \in \mathbb{Z}_l[T]$ be such that $l = \Phi_{l^m}(T) + a(T)(T^{l^{m-1}}-1)$. We recall that $\Phi_{l^m}(T)(T^{l^{m-1}}-1) = T^{l^m}-1$. Therefore the equality $lh(T) = \Phi_{l^m}(T)h(T) + a(T)(T^{l^{m-1}}-1)h(T)$ implies that $lh(T) \in \left(T^{l^m}-1\right)$. Since $\mathbb{Z}_l[T]$ is factorial and $l \nmid T^{l^m}-1$ it follows that $T^{l^m}-1 \mid h(T)$.

In order to prove surjectivity we let (f,g) be such that $f \equiv g (\mathrm{mod}\ (l, T^{l^{m-1}}-1))$. From (21) we deduce that there exist polynomials h_1, h_2 such that

$$f(T) - g(T) = h_1(T)\Phi_{l^m}(T) + h_2(T)(T^{l^{m-1}}-1).$$

Therefore

$$h(T) := f(T) - h_2(T)(T^{l^{m-1}}-1) = g(T) + h_1(T)\Phi_{l^m}(T) \tag{22}$$

maps to (f,g). □

Let

$$f(T) = \sum_{i=0}^{l^{m-1}-1} a_i T^i \text{ and } g(T) = \sum_{j=0}^{l^{m-1}(l-1)-1} b_j T^j$$

and suppose that (f,g) is an element in the pull back. One easily shows that

$$\begin{array}{rcl} & & f(T) \equiv g(T) (\mathrm{mod}\ (l, T^{l^{m-1}}-1)) \\ \iff & & a_i \equiv \displaystyle\sum_{k=0}^{l-2} b_{i+kl^{m-1}} (\mathrm{mod}\ l), \quad i = 0, \ldots, l^{m-1}-1. \end{array} \tag{23}$$

Via the canonical identification $\mathbb{Z}_l[T]/(\Phi_{l^m}(T)) \simeq \mathbb{Z}_l[\zeta_{l^m}]$ the polynomial $g(T)$ corresponds to $\gamma_m = \sum_{j=0}^{l^{m-1}(l-1)-1} b_j \zeta_{l^m}^j$. One has

$$\begin{array}{rcl} & & \gamma_m \equiv \displaystyle\sum_{i=0}^{l^{m-1}-1} a_i \zeta_{l^m}^i (\mathrm{mod}\ (1-\zeta_l)) \\ \iff & & a_i \equiv \displaystyle\sum_{k=0}^{l-2} b_{i+kl^{m-1}} (\mathrm{mod}\ l), \quad i = 0, \ldots, l^{m-1}-1. \end{array} \tag{24}$$

Combining (23) and (24) we obtain

$$f(T) \equiv g(T) (\mathrm{mod}\ (l, T^{l^{m-1}}-1)) \iff \gamma_m \equiv \sum_{i=0}^{l^{m-1}-1} a_i \zeta_{l^m}^i (\mathrm{mod}\ (1-\zeta_l)).$$

We define a homomorphism

$$\varphi_{m-1} \colon \mathbb{Z}_l[C_{l^{m-1}}] \longrightarrow \mathbb{Z}_l[\zeta_{l^m}]/(1-\zeta_l), \quad g_0 \bmod \langle g_0^{l^{m-1}} \rangle \mapsto \zeta_{l^m} \bmod (1-\zeta_l)$$

and are finally in position to formulate our recursive test in terms of congruences. Let $(\gamma_0, \ldots, \gamma_n) \in \mathcal{M}^\times$ and set $\lambda_0 := \gamma_0$. Let $m \in \{1, \ldots, n\}$ and suppose that by

induction we have constructed $\lambda_{m-1} \in \mathbb{Z}_l[C_{l^{m-1}}]$. If $\gamma_m \equiv \varphi_{m-1}(\lambda_{m-1})(\mathrm{mod}\ (1-\zeta_l))$, then $(\gamma_0, \ldots, \gamma_m)$ defines an element $\lambda_m \in \mathbb{Z}_l[C_{l^m}]$ and we can continue the recursive test. Note that λ_m can easily be computed from (22). We summarize our result in the following proposition.

Proposition 5.2. *Let l be a prime and let G be a cyclic group of order l^n. Let $(\gamma_0, \ldots, \gamma_n) \in \mathcal{M}^\times$. Then*

$$(\gamma_0, \ldots, \gamma_n) \in \mathbb{Z}_l[C_{l^n}]^\times \iff \gamma_m \equiv \varphi_{m-1}(\lambda_{m-1})(\mathrm{mod}\ (1-\zeta_l)), \quad m = 1, \ldots, n.$$

Proof. The proof follows immediately from the preceding discussion and the following observation

$$\mathcal{M}^\times \cap \mathbb{Z}_l[G] = \mathbb{Z}_l[G]^\times. \tag{25}$$

To prove (25) let $\varepsilon \in \mathcal{M}^\times \cap \mathbb{Z}_l[G]$ and $\mu \in \mathcal{M}^\times$ such that $\varepsilon\mu = 1$. Since $\mathbb{Z}_l[G]$ is of finite index in $\mathcal{M}$ there exists a natural number n and $a_0, \ldots, a_{n-1} \in \mathbb{Z}_l$ such that $a_0 + a_1\mu + \cdots + a_{n-1}\mu^{n-1} + \mu^n \in \mathbb{Z}_l[G]$. Multiplying by ε^{n-1} shows that $\mu \in \mathbb{Z}_l[G]$. (Note that this proof works in much greater generality.) □

Remark 5.3. a) Let $G = \langle \sigma, \tau \mid \sigma^{l^n} = \tau^2 = 1, \tau\sigma = \sigma^{-1}\tau \rangle$ be the dihedral group of order 2^{l^n}, where l is an odd prime. Then

$$\mathbb{Q}[G] \simeq \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q}(\zeta_l)^+) \oplus \ldots \oplus M_2(\mathbb{Q}(\zeta_{l^n})^+).$$

Let $H = \langle \sigma \rangle$. By [6, Prop. 3.2] we know that the restriction map

$$\mathrm{res}\colon K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors} \longrightarrow K_0(\mathbb{Z}_l[H], \mathbb{Q}_l)_{tors}$$

is injective. Let

$$\alpha = (\alpha_0, \ldots, \alpha_{n+1}) \in \mathbb{Z}_l{}^\times \oplus \mathbb{Z}_l{}^\times \oplus \mathbb{Z}_l[\zeta_l]^{+\times} \oplus \ldots \oplus \mathbb{Z}_l[\zeta_{l^n}]^{+\times}.$$

By [6, Lemma 3.9] or [2, page 575] one has $\mathrm{res}(\alpha) = (\alpha_0\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ and we can apply our recursive test to $\mathrm{res}(\alpha)$ in order to decide whether $\delta_l(\alpha)$ is trivial in $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$.

b) We refer the interested reader to [1, Sec. 6] for numerical examples for dihedral extensions of degree $2l$ where again l denotes an odd prime. Note, however, that in all of these examples the Mordell-Weil group $E(K)$ is finite. For $\mathrm{rk}(E(K)) > 0$ one would have to adapt the approach of Sec. 4 which relies (at least to some extend) on the assumption that G is an l-group.

To conclude this section we compute the order of $K_0(\mathbb{Z}_l[C_{l^n}], \mathbb{Q}_l)_{tors}$.

Proposition 5.4. *Let l be a prime and let G be a cyclic group of order l^n. Then*

$$\#K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors} = (l-1)^n l^e \text{ with } e = \frac{l^n - 1}{l-1} - n.$$

The exponent of $K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors}$ is a divisor of $(l-1)^n l^f$ with $f = \frac{(n-1)n}{2}$

Proof. We set $DT(\mathbb{Z}_l[G]) := K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)_{tors}$. We apply [5, Th. 8.1] to the cartesian square (20). Using the fact that SK_1 of a semilocal commutative ring is trivial (see [14, Th. (45.12)]), we obtain the short exact sequence

$$0 \longrightarrow \mathbb{F}_l[C_{l^{n-1}}]^\times \longrightarrow DT(\mathbb{Z}_l[C_{l^n}]) \longrightarrow DT(\mathbb{Z}_l[C_{l^{n-1}}]) \longrightarrow 0$$

For a natural number k the ring $\mathbb{F}_l[C_{l^k}]$ is local with maximal ideal $\Delta := \ker(aug)$, where $aug : \mathbb{F}_l[C_{l^k}] \longrightarrow \mathbb{F}_l$ is the usual augmentation map. It follows that $\#\mathbb{F}_l[C_{l^k}]^\times = l^{l^k-1}(l-1)$.

The result for the order of $K_0(\mathbb{Z}_l[G],\mathbb{Q}_l)_{tors}$ follows now easily by induction. It is also easily seen that

$$\prod_{k=0}^{n-1}\exp(\mathbb{F}_l[C_{l^k}]^\times).$$

annihilates $DT(\mathbb{Z}_l[G])$. Finally, from $\exp(\mathbb{F}_l[C_{l^k}]^\times)=l^k(l-1)$ we obtain the result for the exponent of $DT(\mathbb{Z}_l[G])$. □

Remark 5.5. a) Let l be an odd prime and let G be a cyclic group of order l^2. Then it can be shown that

$$K_0(\mathbb{Z}_l[G],\mathbb{Q}_l)_{tors}\simeq C_{l-1}^2\times C_l^{l-1}.$$

Of course, one would like to have a general result which describes the structure of $K_0(\mathbb{Z}_l[G],\mathbb{Q}_l)_{tors}$ for cyclic groups of (odd) prime power order. However, our proof is purely computational and although several parts obviously generalize, it finally becomes a mess for $n>2$.

b) Running the algorithm of [5] one obtains

$$K_0(\mathbb{Z}_3[C_{27}],\mathbb{Q}_3)_{tors}\simeq C_2^3\times C_3^4\times C_9^3.$$

For higher values of l and n the algorithm does not terminate.

6. Numerical results

Let l be an odd prime, $n\geq 1$ a natural number and p a prime such that $p\equiv 1(\mathrm{mod}\ l^n)$. Let K denote the unique subextension of $\mathbb{Q}(\zeta_p)$ of order l^n. We did many experiments with various elliptic curves from Cremona's database each time verifying ETNC_l numerically modulo $\mathcal{E}$. We point out once again that we only provide numerical evidence for the rationality conjecture since we only compute complex approximations to the L-values, regulators and periods. Moreover, we are only able to compute a conjectural value for the order of $Ш(E/K)$ from the classical BSD conjecture for E/K.

Assuming the rationality conjecture and that

- we have correctly computed the exact values η_i from (19) by some rounding process,
- the value for the order of $Ш(E/K)$ is correct,

the remaining computations are exact. We point out, that presently in none of the computed examples we actually have a rigorous proof.

The MAGMA implementation, sample files and two tables are available from

`http://www.mathematik.uni-kassel.de/~bley/pub.html`.

We have produced two tables of examples. Table 1 contains a list of examples as described above where we have checked the ETNC at l. More precisely, we considered all elliptic curves with split multiplicative reduction of conductor $N_E\leq 500$, primes $p\leq 50$ and triples

$$\begin{aligned}(r,l,n)\quad\in\quad&\{[0,3,1],[0,3,2],[0,5,1],[0,7,1],[0,11,1],\\&[1,3,1],[1,3,2],[1,5,1],[1,7,1],[1,11,1],\\&[2,3,1],[2,3,2],[2,5,1],[2,7,1],[2,11,1]\}\end{aligned}$$

such that the pair (E,K) satisfies our Hypothesis (i)-(vi). The table contains in total 1507 examples.

If $r = 0$ we can fully verify the ETNC_l numerically and, in addition, the methods of [1] are available. Also note that by [17, Th. 3.3 and 3.5] we know that if the analytic rank is trivial then $E(K)$ and $Ш(E/K)$ are finite. Note also that the theory of modular symbols would allow to compute the precise value of the L-series (see also Remark 4.6). However, this is not implemented. In Table 2 we list all examples as described above where we tried to check the full ETNC. More precisely, we considered all elliptic curves with split multiplicative reduction of conductor $N_E \leq 100$, primes $p \leq 50$ and triples

$$(r, l, n) \in \{[0, 3, 1], [0, 3, 2], [0, 5, 1], [0, 7, 1], [0, 11, 1], \}.$$

such that the pair (E, K) satisfies our Hypothesis (i)-(viii). Each of the examples is followed by a set of primes which contains the primes where we could not apply the methods of [1] because the Hypothesis (H0)-(H5) were not satisfied. In all cases this set consist of at most 3 primes. The table contains in total 208 examples. Note that for 52 examples we obtained a numerical verification of ETNC at all primes.

We describe one of these examples in detail. We let E be the curve $11A3$, $l = 3, p = 7$ and $n = 1$. So K is the cubic extension of conductor 7. We have $N_E = 11$ and $d_{K/\mathbb{Q}} = 49$.

We have three characters

	id	g_0	g_0^2
χ_1	1	1	1
χ_2	1	ζ_3	ζ_3^2
χ_3	1	ζ_3^2	ζ_3

where we identify ζ_3 with $\exp(2\pi i/3)$. Hence $\mathbb{Q}[G] \simeq \mathbb{Q} \oplus \mathbb{Q}(\zeta_3)$. Elements in the center of $\mathbb{C}[G]$ will be denoted by 3-tuples $z = (z_1, z_2, z_3), z_i \in \mathbb{C}$. Recall that $z \in \zeta(\mathbb{Q}[G])$ if and only if $z_1 \in \mathbb{Q}$, $z_2, z_3 \in \mathbb{Q}(\zeta_3)$ and $\sigma(z_2) = z_3$, where $\sigma(\zeta_3) = \zeta_3^2$. Elements in $z \in \zeta(\mathbb{Q}[G])$ will be represented by tuples $z = (z_1, z_2)$.

The L-values were computed with a precision of 20 decimal digits and are given by

$$\begin{aligned}(L(E/\mathbb{Q}, \bar{\chi}, 1))_{\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)} = (\quad & 0.25384186085591068434, \\ & 1.9971068270600871687 + 1.328439293785575938 21i, \\ & 1.9971068270600871687 - 1.328439293785575938 21i \quad).\end{aligned}$$

For the resolvents $R = R(\alpha_0)$ with respect to the integral normal basis element $\alpha_0 = \mathrm{Tr}_{\mathbb{Q}(\zeta_7)/K}(\zeta_7)$ we computed

$$\begin{aligned}(R(\alpha_0))_{\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)} = (\quad & -1.0000000000000000000, \\ & 2.3704694055762005916 + 1.1751062918847870026i, \\ & 2.3704694055762005916 - 1.1751062918847870026i \quad)\end{aligned}$$

and, finally, for the periods we obtain

$$\begin{aligned}(\Omega_\chi)_{\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)} = (\quad & 0.15757842250733170863, \\ & 0.15757842250733170863, \\ & 0.15757842250733170863 \quad).\end{aligned}$$

The analytic rank of each of the twisted L-functions is therefore 0 and we conclude from the theorem of Longo and Tian-Zhang (see [17, Th. 3.7]) that $E(K)$ is finite.

The equivariant BSD-quotient $u = \mathcal{L}^* R/\Omega$ is given by

$$\begin{aligned} u = (\quad & -0.040000000000000000000, \\ & 0.500000000000000000002 + 0.866025403784438646 78i, \\ & 0.500000000000000000002 - 0.866025403784438646 78i \quad). \end{aligned}$$

Numerically this confirms the rationality conjecture because u is close to

$$\left(\frac{-1}{25}, \zeta_3 + 1, -\zeta_3\right)$$

and $\sigma(-\zeta_3) = -\zeta_3^2 = \zeta_3 + 1$. The Euler factor at $p = 7$ equals $(\frac{10}{7}, 1, 1)$, so that $\xi_3 = (\frac{7}{10}, 1, 1)$. Finally we obtain $u_l \xi_l^{-1} = (\frac{-2}{35}, \zeta_3 + 1)$ and one checks that $\frac{-2}{35} \equiv \zeta_3 + 1 (\mathrm{mod}\ 1 - \zeta_3)$, so that numerically $ETNC_l$ is correct.

We briefly recall the main results of [1], namely Proposition 4.4 and Corollary 4.7. We let henceforth l denote an arbitrary prime. Then there is a finite set of difficult primes HP such that for all primes $l \notin HP$ one has

$$\mathrm{ETNC}_l \text{ holds } \iff u \text{ has support in } HP.$$

We recall the definition of HP. Let $c_v(E_K)$ denote the Tamagawa number for a finite place v of K and write $S_l(K)$ for the places of K lying over places in $S_l = S \cup \{l\}$. Then one has

$$\begin{aligned} HP \quad = \quad & S \cup \{2\} \cup \{l \colon l \mid \#G\} \cup \{l \colon l \mid c_v(E_K) \text{ for a } v \in S_l(K)\} \\ & \{l : l \mid \#E(K)_{tors}\} \cup \{l : l \mid \#Ш(E/K)\}. \end{aligned}$$

We obviously have $S = \{7, 11\}$, $c_{v_{11}}(E) = 1$ for the unique place v_{11} of K above 11, $\#E(K)_{tors} = 5$ and conjecturally (computed from BSD for E/K) one has $\#Ш(E/K) = 1$. Therefore $HP = \{2, 3, 5, 7, 11\}$. We have already checked $ETNC_3$ and for all the other primes $l \in HP$ we are able to verify $ETNC_l$ using [1, Prop. 4.4]. This is possible since, firstly, for all of these primes (H0)-(H5) are satisfied, and secondly, we are able to perform all the necessary computations.

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