GEOMETRIC MAIN CONJECTURES IN FUNCTION FIELDS

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ABSTRACT. We prove an Equivariant Main Conjecture in Iwasawa Theory along any rank one, sign-normalized Drinfeld modular, split at ∞ Iwasawa tower of a general function field of characteristic p, for the Iwasawa modules recently considered by Greither and Popescu in [5], in their proof of the classical Equivariant Main Conjecture along the (arithmetic) cyclotomic Iwasawa tower. As a consequence, we prove an Equivariant Main Conjecture for a projective limit of certain Ritter–Weiss type modules, along the same Drinfeld modular Iwasawa towers. This generalizes the results of Anglès et.al. [1], Bandini et al. [2], and Coscelli [3], for the split at ∞ piece of the Iwasawa towers considered in loc.cit., and refines the results in [5].

1. INTRODUCTION AND NOTATIONS

1.1. **Arithmetic Iwaswa Theory.** In [5], Greither and the second author considered a set of data $(K/k, S, \Sigma)$ consisting of an abelian extension K/k of global fields of characteristic $p > 0$ of Galois group G and two finite, nonempty, disjoint sets of places S and Σ in k, such that S contains the ramification locus of K/k. From this data one can construct a Deligne–Picard 1–motive $M_{S,\Sigma}$, which is naturally acted upon by the Galois group $G \times \mathbf{\Gamma}$, where $\mathbf{\Gamma} := G(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ and $\mathbb{F}_q$ is the exact field of constants of k. As a consequence, all the ℓ–adic realizations $T_\ell(M_{S,\Sigma})$ are natural finitely generated modules over the profinite group–algebra $\mathbb{Z}_\ell[[G \times \mathbf{\Gamma}]]$, for all prime numbers ℓ.

On the other hand, the set of data $(K/k, S, \Sigma)$ gives rise to a polynomial $\Theta_{S,\Sigma}(u) \in \mathbb{Z}[G][u]$, which is uniquely determined by the packet of (S–incomplete, Σ–smoothed) Artin L–functions $L_{S,\Sigma}(\chi, s)$, for all the complex valued characters χ of G, via the equalities

$$\chi(\Theta_{S,\Sigma}(u)) \mid_{u = q^{-s}} = L_{S,\Sigma}(\chi^{-1}, s),$$

for all $s \in \mathbb{C}$. The main result in [5] is the following G–equivariant Iwasawa main conjecture, along the arithmetic (cyclotomic) Iwasawa tower $(K \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q})/K$, of Galois group $\Gamma \simeq \widehat{\mathbb{Z}}$, whose natural topologial generator is the q–power Frobenius automorphism of $\overline{\mathbb{F}_q}$, denoted by γ.

Theorem 1.1 (Greither–Popescu [5])**.** *For $(K/k, S, \Sigma)$ as above and all primes ℓ we have*
  (1) $\mathrm{pd}_{\mathbb{Z}_\ell[[G \times \Gamma]]}(T_\ell(M_{S,\Sigma})) = 1.$
  (2) $\mathrm{Fitt}_{\mathbb{Z}_\ell[[G \times \Gamma]]}(T_\ell(M_{S,\Sigma})) = \langle \Theta_{S,\Sigma}(\gamma^{-1}) \rangle.$

In the statement above, $\mathrm{pd}_R(M)$ and $\mathrm{Fitt}_R(M)$ denote the projective dimension, respectively the 0–th Fitting ideal of a finitely presented module M over a commutative, unital ring R. See §4 in [5] for the relevant definitions and properties of Fitting ideals.

1.2. **Geometric Iwasawa Theory.** The main goal of this paper is to prove analogues of Theorem 1.1 above along geometric Iwasawa–towers K_∞/K, which are highly ramified and obtained from K essentially by adjoining the $\mathfrak{p}^n$–torsion points of a sign–normalized, rank one Drinfeld module (a Hayes module), for some place $\mathfrak{p}$ in k and all $n \in \mathbb{Z}_{\geq 1}$.

This geometric Iwasawa theoretic approach was first considered in [1], in the particular case where $K = k = \mathbb{F}_q(t)$ and $K_\infty = \cup_{n \geq 0} K_n$ with K_n obtained by adjoining the $\mathfrak{p}^{n+1}$–torsion $\mathcal{C}[\mathfrak{p}^{n+1}]$ of the Carlitz module

$$\mathcal{C} : \mathbb{F}_q[t] \to \mathbb{F}_q[t]\{\tau\}, \qquad \mathcal{C}(t) = t + \tau,$$

for a maximal ideal $\mathfrak{p}$ in $\mathbb{F}_q[t]$. The fields K_n are the ray–class fields of k of conductors $(\mathfrak{p}^{n+1}v_\infty)$, where v_∞ is the valuation of k of uniformizer $1/t$. While using Theorem 1.1 above and the techniques and results developed in [5], the authors of [1] are studying the more classical Iwasawa $\mathbb{Z}_p[[G(K_\infty/k)]]$–module

$$\mathfrak{X}_p^{(\infty)} := \varprojlim_n (\mathrm{Pic}^0(K_n) \otimes \mathbb{Z}_p),$$

where the projective limit is taken with respect to the usual norm maps at the level of the Picard groups of the function fields K_n. One has topological group isomorphisms

$$G(K_\infty/K) \simeq \mathbb{F}_\mathfrak{p}^\times \times U_\mathfrak{p}^{(1)} \simeq \mathbb{F}_\mathfrak{p}^\times \times \mathbb{Z}_p^{\aleph_0},$$

where $\mathbb{F}_\mathfrak{p}$ is the residue field of $\mathfrak{p}$, $U_\mathfrak{p}^{(1)}$ is the group of principal units in the completion of k at $\mathfrak{p}$ and $\mathbb{Z}_p^{\aleph_0}$ denotes a product of countably many copies of $\mathbb{Z}_p$. The main Iwasawa theoretic result in [1] gives the 0–th Fitting ideal of $\mathfrak{X}_p^\infty$, away from the trivial character of $\mathbb{F}_\mathfrak{p}^\times$, in terms of an element $\Theta_{S,\Sigma}^{\infty,\sharp} \in \mathbb{Z}_p[[G(K_\infty/K)]]$, which should be viewed as the $\mathbb{Z}_p[[G(K_\infty/k)]]$–analogue of the special value $\Theta_{S,\Sigma}(1) \in \mathbb{Z}[G(K/k)]$ of the element $\Theta_{S,\Sigma}(u)$ described above. The work in [1] was further developed in [2] and [3], see Remark 3.18.

As opposed to [1], the set-up of this paper is the following. We fix an arbitrary function field k of exact field of constants $\mathbb{F}_q$ and a place v_∞ of k, called the infinite place of k from now on. We let A denote the Dedekind domain consisting of those elements in k which are integral at all places of k, except for v_∞. Further, we fix an ideal $\mathfrak{f}$ and a maximal ideal $\mathfrak{p}$ of A, such that $\mathfrak{p} \nmid \mathfrak{f}$. The geometric extensions of k of interest to us are the fields

$$L_n := H_{\mathfrak{f}\mathfrak{p}^{n+1}}, \text{ for all } n \geq 0,$$

which are the ray–class fields of k of conductors $\mathfrak{f}\mathfrak{p}^{n+1}$ in which v_∞ splits completely (i.e. the *real* ray–class fields of conductors $\mathfrak{f}\mathfrak{p}^{n+1}$.) As proved by Hayes in [7], the extension L_n/L_0 is essentially generated by the $\mathfrak{p}^{n+1}$–torsion points of a certain type of rank 1, sign-normalized Drinfeld module defined on A. (See Section 2.1 for details.) The ensuing geometric Iwasawa tower L_∞/k, with $L_\infty = \cup_n L_n$, has Galois group G_∞ which sits in an exact sequence

$$0 \to G(L_\infty/L_0) \to G_\infty \to G(L_0/k) \to 0,$$

where $G(L_\infty/L_0) \simeq \mathbb{Z}_p^{\aleph_0}$ and $G(L_0/k)$ finite. Since the ramification locus of L_∞/k is finite, namely $S := \{\mathfrak{p}\} \cup \{v \mid v \text{ prime in } A, v|\mathfrak{f}\}$, one can construct the following element

$$\Theta_{S,\Sigma}^{(\infty)}(u) := \varprojlim_n \Theta_{S,\Sigma}^{(n)}(u) \in \mathbb{Z}_p[[G_\infty]][[u]],$$

out of the polynomials $\Theta_{S,\Sigma}^{(n)}(u) \in \mathbb{Z}[G(L_n/k)][u]$ associated in [5, §4.2] to the data $(L_n/k, S, \Sigma)$, for any finite, non-empty set Σ of primes in k, disjoint from S. On the other hand, to the set of data $(L_\infty/k, S, \Sigma)$ one can associate the following $\mathbb{Z}_p[[G_\infty \times \Gamma]]$–module

$$T_p(M_{S,\Sigma}^{(\infty)}) := \varprojlim_n T_p(M_{S,\Sigma}^{(n)}),$$

where $M_{S,\Sigma}^{(n)}$ is the Picard 1–motive for $(L_n/k, S, \Sigma)$, and $T_p(M_{S,\Sigma}^{(n)})$ is its p–adic Tate module, as defined in [5, §2]. The projective limit is taken with respect to certain canonical norm maps, described in detail in §3 below. It turns out that neither $T_p(M_{S,\Sigma}^{(n)})$ nor $T_p(M_{S,\Sigma}^{(\infty)})$ depend on Σ, reason for which we will drop Σ from those notations. In §3.2, we prove the following geometric–arithmetic analogue of Theorem 1.1 above.

Theorem 1.2. *For any finite, non-empty set Σ of primes in k, disjoint from S, the $\mathbb{Z}_p[[G_\infty \times \Gamma]]$–module $T_p(M_S^{(\infty)})$ is finitely generated, torsion and*

(1) $\mathrm{pd}_{\mathbb{Z}_p[[G_\infty \times \Gamma]]}(T_p(M_S^{(\infty)})) = 1$.
(2) $\mathrm{Fitt}_{\mathbb{Z}_p[[G_\infty \times \Gamma]]}(T_p(M_S^{(\infty)})) = \langle \Theta_{S,\Sigma}^{(\infty)}(\gamma^{-1}) \rangle$.

In order to obtain a geometric (along the tower L_∞/k) Iwasawa main conjecture–type result, one has to take Γ–coinvariants. In §3.3 we establish a $\mathbb{Z}_p[[G_\infty]]$–module isomorphism

$$T_p(M_S^{(\infty)})_\Gamma \simeq \bigtriangledown_S^{(\infty)},$$

where $\bigtriangledown_S^{(\infty)}$ is an arithmetically meaningful $\mathbb{Z}_p[[G_\infty]]$–module, a projective limit of Ritter–Weiss type modules $\nabla_S^{(n)}$ which are extensions of divisor groups by class groups. (See the Appendix. Also, see [15] for the number field analogues of $\bigtriangledown_S^{(n)}$.) We prove the following.

Theorem 1.3. *For any finite, non-empty set Σ of primes in k, disjoint from S, the $\mathbb{Z}_p[[G_\infty]]$–module $\bigtriangledown_S^{(\infty)}$ is finitely generated, torsion and*

(1) $\mathrm{pd}_{\mathbb{Z}_p[[G_\infty]]}(\bigtriangledown_S^{(\infty)}) = 1$.
(2) $\mathrm{Fitt}_{\mathbb{Z}_p[[G_\infty]]}(\bigtriangledown_S^{(\infty)}) = \langle \Theta_{S,\Sigma}^{(\infty)}(1) \rangle$.

To relate our results to those in [1, 2, 3] we have to introduce some further notation. We let Δ denote the maximal subgroup of $G(L_0/k)$ whose order is not divisible by p. Then we have a canonical isomorphism $G_\infty \simeq \Delta \times G_\infty^{(p)}$ where $G_\infty^{(p)}$ is the maximal pro-p subgroup of G_∞. We view the idempotent $e_\Delta := \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \delta$ as an element of $\mathbb{Z}_p[[G_\infty]]$ and consider the exact functor $M \mapsto M^\sharp := (1 - e_\Delta)M$ from the category of $\mathbb{Z}_p[[G_\infty]]$-modules to the category of modules over the quotient ring $\mathbb{Z}_p[[G_\infty]]^\sharp := (1 - e_\Delta)\mathbb{Z}_p[[G_\infty]]$.

Further, if $\mathfrak{f}$ is the unit ideal $\mathfrak{e}$ and the prime $\mathfrak{p}$ stays inert in the real Hilbert class field $H_\mathfrak{e}$ over k, then for $S = \{\mathfrak{p}\}$ one has an isomorphism of $\mathbb{Z}_p[[G_\infty]]^\sharp$–modules (see §3.3 below)

$$\bigtriangledown_S^{(\infty),\sharp} \simeq \mathfrak{X}_p^{(\infty),\sharp}.$$

Since these additional hypotheses are obviously satisfied when $k = \mathbb{F}_q(t)$ (as $H_\mathfrak{e} = k$ in that case), Theorem 1.3 above implies the main Iwasawa theoretic result in [1] discussed above, for the real Carlitz tower, i.e the maximal subfield of K_∞ where v_∞ splits completely. (See

Theorem 3.17.) Under slightly stronger hypotheses, we obtain an isomorphism of $\mathbb{Z}_p[[G_\infty]]$–modules $\bigtriangledown_S^{(\infty)} \simeq \mathfrak{X}_p^{(\infty)}$ which leads to a full description of $\mathrm{Fitt}_{\mathbb{Z}_p[[G_\infty]]}(\mathfrak{X}_p^{(\infty)})$. (See Theorem 3.16.) For an even more detailed comparison of our results with those in [1, 2, 3] we refer the reader to Remark 3.18 below.

In order to establish the link between the modules $\bigtriangledown_S^{(\infty)}$ and $\mathfrak{X}_p^{(\infty)}$ mentioned above, we needed to provide slight generalizations of the results in [4] on Ritter-Weiss modules and Tate sequences for function fields. This is done in the Appendix.

Remark 1.4. Unlike in classical Iwasawa theory, all Iwasawa algebras $\mathbb{Z}_p[[G_\infty \times \Gamma]]$, $\mathbb{Z}_p[[\Gamma]]$ and $\mathbb{Z}_p[[G_\infty]]$ relevant in this context are not Noetherian. In particular, one has an isomorphism

$$\mathbb{Z}_p[[G_\infty]] \simeq \mathbb{Z}_p[t(G_\infty)][[X_1, X_2, \dots]]$$

of topological rings, where the power series ring has countably many variables and $t(G_\infty)$ is the (finite) torsion subgroup of G_∞. Throughout, if R is a commutative ring, the ring of power series in countably many variables with coefficients in R is defined by

$$R[[X_1, X_2, \dots]] := \varprojlim_n R[[X_1, X_2, \dots, X_n]],$$

where the transition maps $R[[X_1, X_2, \dots, X_{n+1}]] \to R[[X_1, X_2, \dots, X_n]]$ are the R–algebra morphisms sending $X_i \mapsto X_i$, for all $i \leq n$, and $X_{n+1} \mapsto 0$.

2. Class–field theory and geometric Iwasawa towers

2.1. **Sign normalized Drinfeld modules and class field theory.**
This subsection follows the exposition of Hayes [7]. In particular, we recall without proof the results of §4 of loc.cit.

Let k be a global function field and let $\mathbb{F}_q$ be its field of constants. For a place (discrete, rank 1 valuation) v of k, we let k_v denote the completion of k in the v–adic topology. We let $\mathcal{O}_{k_v}$, $\mathfrak{m}_{k_v}$, U_{k_v} be the ring of integers in k_v, its maximal ideal, and its group of units, respectively. As usual, we let $U_{k_v}^{(n)} := 1 + \mathfrak{m}_{k_v}^n$, for all $n \geq 1$. We denote by d_v the degree of v relative to $\mathbb{F}_q$ and by $\mathbb{F}_v$ its residue field. By definition, we have $\mathbb{F}_v = \mathbb{F}_{q^{d_v}}$. Further, we fix a uniformiser $\pi_v \in k$, for all v as above. For every v, we have group isomorphisms

$$k_v^\times \simeq \pi_v^{\mathbb{Z}} \times U_{k_v}, \qquad U_{k_v} \simeq \mathbb{F}_v^\times \times U_{k_v}^{(1)}.$$

Now, we fix once and for all a place v_∞ of k (called the place at infinity) and let A be the Dedekind ring of elements of k that are integral outside v_∞. (In [7] our A is denoted by A_∞.) Note that the places v of k, which are different from v_∞, are in one–to–one correspondence with the maximal ideals of A. For such a maximal ideal $\mathfrak{p}$, we denote by $v_\mathfrak{p}$ the corresponding place of k, viewed as a rank one, discrete valuation of k, normalized so that $v_\mathfrak{p}(k^\times) = \mathbb{Z}$.

In order to simplify notations, we let $k_\infty = k_{v_\infty}$, $\pi_\infty := \pi_{v_\infty}$, $\mathbb{F}_\infty := \mathbb{F}_{v_\infty}$, $d_\infty := d_{v_\infty}$, etc. The same notation principle applies to the finite places $v_\mathfrak{p}$, namely $k_\mathfrak{p} := k_{v_\mathfrak{p}}$, etc. Finally, for any place v, we let $\mathrm{ord}_v : k_v^\times \to \mathbb{Z}$ be its associated valuation, normalized so that $\mathrm{ord}_v(\pi_v) = 1$.

Let I_A denote the group of fractional ideals of A and let P_A be the subgroup of principal ideals. Then $\mathrm{Pic}(A) = I_A/P_A$ and we write $h_A := |\mathrm{Pic}(A)|$ for the class number of A. We write D_k for the group of divisors of k and D_k^0 for the subgroup of divisors of degree zero. Let

div $\colon k^{\times} \longrightarrow D_k^0$ be the divisor map and set $\mathrm{Pic}^0(k) := D_k^0/\mathrm{div}(k^{\times})$. We write $h_k := |\mathrm{Pic}^0(k)|$. Recall that we have an exact sequence

$$0 \to \mathrm{Pic}^0(k) \to \mathrm{Pic}(A) \xrightarrow{\widehat{\deg}} \mathbb{Z}/d_{\infty} \to 0,$$

where $\widehat{\deg}$ is the degree modulo d_{∞} map. Consequently, we have an equality $h_A = h_k d_{\infty}$.

Definition 2.1. As in [7], we define the following.
   (1) A finite, Galois extension K/k is called real (relative to v_{∞}) if v_{∞} splits completely in K/k, or, equivalently, if there exists a k–embedding $K \hookrightarrow k_{\infty}$.
   (2) For an integral ideal $\mathfrak{m} \subseteq A$, $H_{\mathfrak{m}}$ denotes the real ray–class field of k of conductor $\mathfrak{m}$.
   (3) If $\mathfrak{m} = \mathfrak{e} := A$ is the unit ideal, then we call $H_{\mathfrak{e}}$ the real Hilbert class field of k.

Next, we give the idèle theoretic description of the class fields $H_{\mathfrak{m}}$, as in [7]. For that, let J_k denote the group of idèles of k and consider the following subgroups of J_k.

$$U(\mathfrak{m}) := k_{\infty}^{\times} \times \prod_{\mathfrak{p} | \mathfrak{m}} U_{\mathfrak{p}}^{(v_{\mathfrak{p}}(\mathfrak{m}))} \times \prod_{\mathfrak{p} \nmid \mathfrak{m}\infty} U_{\mathfrak{p}}, \qquad J_{\mathfrak{m}} := k^{\times} \cdot U(\mathfrak{m}).$$

The following is proved in [7].

Proposition 2.2. *For all $\mathfrak{m}$ as above, the Artin reciprocity map gives group isomorphisms*

$$J_k/J_{\mathfrak{m}} \simeq G(H_{\mathfrak{m}}/k), \qquad J_k/J_{\mathfrak{e}} \simeq G(H_{\mathfrak{e}}/H), \qquad J_{\mathfrak{e}}/J_{\mathfrak{m}} \simeq G(H_{\mathfrak{m}}/H_{\mathfrak{e}}).$$

Further, if $\mathfrak{m} \neq \mathfrak{e}$, we have canonical group isomorphisms

$$J_k/J_{\mathfrak{e}} \simeq \mathrm{Pic}(A), \qquad J_{\mathfrak{e}}/J_{\mathfrak{m}} \simeq (A/\mathfrak{m})^{\times}/\mathbb{F}_q^{\times}.$$

The real ray–class fields $H_{\mathfrak{m}}$ are contained in slightly larger abelian extensions $H_{\mathfrak{m}}^*/k$, of conductor $\mathfrak{m} \cdot v_{\infty}$, tamely ramified at v_{∞}. The advantage of passing from $H_{\mathfrak{m}}$ to $H_{\mathfrak{m}}^*$ is that the latter can be explicitly constructed by adjoining torsion points of certain rank 1 A–Drinfeld modules to the field of definition $H_{\mathfrak{e}}^*$ of these Drinfeld modules. Next, we give the idèle theoretic description and explicit construction of the fields $H_{\mathfrak{m}}^*$, both due to Hayes [7].

As in [7], let us fix a sign function

$$\mathrm{sgn} : k_{\infty}^{\times} \to \mathbb{F}_{\infty}^{\times}.$$

By definition, this is a group morphism, such that $\mathrm{sgn}(U_{\infty}^{(1)}) = 1$ and $\mathrm{sgn}|_{\mathbb{F}_{\infty}^{\times}} = \mathrm{id}_{\mathbb{F}_{\infty}^{\times}}$. Note that sgn is uniquely determined by the value $\mathrm{sgn}(\pi_{\infty})$ at the fixed uniformiser π_{∞}.

For every integral ideal $\mathfrak{m} \subseteq A$, we define the following subgroups of J_k.

$$U^*(\mathfrak{m}) := \{(\alpha_v)_v \in U(\mathfrak{m}) \mid \mathrm{sgn}(\alpha_{\infty}) = 1\}, \quad J_{\mathfrak{m}}^* := k^{\times} \cdot U^*(\mathfrak{m}).$$

Definition 2.3. For all integral ideals $\mathfrak{m} \subseteq A$, we define $H_{\mathfrak{m}}^*$ to be the unique abelian extension of k which corresponds to the subgroup $J_{\mathfrak{m}}^*$ of J_k via the standard class–field theoretic correspondence.

For all $\mathfrak{m}$ as above, with $\mathfrak{m} \neq \mathfrak{e}$, we have a canonical commutative diagram of group morphisms with exact rows and columns.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{F}_q^\times & \longrightarrow & \mathbb{F}_\infty^\times & \longrightarrow & \mathbb{F}_\infty^\times/\mathbb{F}_q^\times & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (A/\mathfrak{m})^\times & \longrightarrow & J_k/J_\mathfrak{m}^* & \longrightarrow & J_k/J_\mathfrak{e}^* & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (A/\mathfrak{m})^\times/\mathbb{F}_q^\times & \longrightarrow & J_k/J_\mathfrak{m} & \longrightarrow & J_k/J_\mathfrak{e} & \longrightarrow & 0
\end{array}
$$

As a consequence, we have the following diagram of abelian extensions of k, whose relative Galois groups are canonically isomorphic to the labels on the connecting line segments.

(1)

The extensions $H_\mathfrak{e}^*/H_\mathfrak{e}$ and $H_\mathfrak{m}^*/H_\mathfrak{m}$ are totally and tamely ramified at the primes above v_∞.

Next, we describe the explicit Drinfeld modular construction of the fields $H_\mathfrak{e}^*$ and $H_\mathfrak{m}^*$, for all $\mathfrak{m} \neq \mathfrak{e}$. Let $\mathbb{C}_\infty$ be the v_∞–completion of the algebraic closure of k_∞. Let $\mathbb{C}_\infty\{\tau\}$ be the non-commutative ring of twisted polynomials with the rule $\tau\omega = \omega^q\tau$, for $\omega \in \mathbb{C}_\infty$. We write

$$
D \colon \mathbb{C}_\infty\{\tau\} \longrightarrow \mathbb{C}_\infty, \quad a_0\tau^0 + a_1\tau^1 + \ldots + a_d\tau^d \mapsto a_0,
$$

for the constant term map.

Definition 2.4 (Hayes). A map $\rho \colon A \longrightarrow \mathbb{C}_\infty\{\tau\}$, $x \mapsto \rho_x$, is called a sgn–normalized Drinfeld module of rank one if the following are satisfied.

(a) ρ is an $\mathbb{F}_q$-algebra homomorphism.
(b) $\deg_\tau(\rho_x) = \deg(x) := \dim_{\mathbb{F}_q}(A/xA) = -\mathrm{ord}_{v_\infty}(x)d_\infty$, for all $x \in A$.
(c) The map $A \longrightarrow \mathbb{C}_\infty$, $x \mapsto D(\rho_x)$, is the inclusion $A \subseteq \mathbb{C}_\infty$.
(d) If $s_\rho(x)$ denotes the leading coefficient of $\rho_x \in \mathbb{C}_\infty\{\tau\}$, then $s_\rho(x) \in \mathbb{F}_\infty^\times$, for all $x \in A$.

(e) If one extends s_ρ to $k_\infty^\times$ by sending

$$x = \sum_{i=i_0}^\infty a_i \pi_\infty^i \mapsto a_{i_0} s_\rho(\pi_\infty)^{i_0},$$

if $a_{i_0} \neq 0$, then $s_\rho : k_\infty^\times \longrightarrow \mathbb{F}_\infty^\times$ is a twist of sgn, i.e there exists $\sigma \in \mathrm{G}(\mathbb{F}_\infty/\mathbb{F}_q)$, such that $s_\rho = \sigma \circ \mathrm{sgn}$.

Any Drinfeld module as above endows $(\mathbb{C}_\infty, +)$ with an A–module structure given by

$$x * z := \rho_x(z), \qquad \text{for all } x \in A,\ z \in \mathbb{C}_\infty,$$

where $(\sum_i a_i \tau^i)(z) = \sum_i a_i z^{q^i}$.

Definition 2.5. Let $\rho : A \to \mathbb{C}_\infty\{\tau\}$ be a rank 1, sgn–normalized Drinfeld module as above.
   (a) The minimal field of definition k_ρ of ρ is the extension of k inside $\mathbb{C}_\infty$ generated by the coefficients of the twisted polynomials ρ_x, for all $x \in A$.
   (b) For all integral ideals $\mathfrak{m} \subseteq A$ with $\mathfrak{m} \neq \mathfrak{e}$, we let

$$\rho[\mathfrak{m}] := \{\alpha \in \mathbb{C}_\infty \mid \rho_x(\alpha) = 0 \text{ for all } x \in \mathfrak{m}\}$$

denote the A–module of $\mathfrak{m}$–torsion points of ρ.

The following gives an explicit construction of the class–fields $H_\mathfrak{m}^*$. (See [7, §4] for proofs.)

Proposition 2.6 (Hayes [7]). *Let ρ be a rank 1, sgn–normalized Drinfeld module as above. Then, the following hold, for all ideals $\mathfrak{m} \subseteq A$, with $\mathfrak{m} \neq \mathfrak{e}$.*
   (1) *The minimal field of definition k_ρ of ρ equals $H_\mathfrak{e}^*$.*
   (2) *We have an equality $H_\mathfrak{m}^* = H_\mathfrak{e}^*(\rho[\mathfrak{m}])$.*
   (3) *The $A/\mathfrak{m}$–module $\rho[\mathfrak{m}]$ is free of rank 1 and, via the canonical isomorphism*

$$(A/\mathfrak{m})^\times \simeq G(H_\mathfrak{m}^*/H_\mathfrak{e}^*), \qquad \widehat{x} \to \sigma_{\widehat{x}},$$

we have $\sigma_{\widehat{x}}(\alpha) = s_\rho(x)^{-1} \cdot \rho_x(\alpha)$, for all $x \in A$ coprime to $\mathfrak{m}$ and all $\alpha \in \rho[\mathfrak{m}]$.

Remark 2.7. The proposition above should be viewed as the function field analogue of the theory of complex multiplication for quadratic imaginary fields, where the role of ρ is played by an elliptic curve with CM by the ring of integers of a quadratic imaginary field k.

2.2. $\mathbb{Z}_p^{\aleph_0}$–extensions (Geometric Iwasawa towers).
We continue to use the notation of Subsection 2.1. We fix a prime ideal $\mathfrak{p}$ of A and an integral ideal $\mathfrak{f} \subseteq A$ which is coprime to $\mathfrak{p}$. For all $n \geq 0$, we consider the following abelian extensions of k, viewed as subfields of $\mathbb{C}_\infty$:

$$L_n := H_{\mathfrak{f}\mathfrak{p}^{n+1}}, \qquad L_n^* := H_{\mathfrak{f}\mathfrak{p}^{n+1}}^*$$

and set

$$L_\infty := \bigcup_{n \geq 0} L_n, \qquad L_\infty^* := \bigcup_{n \geq 0} L_n^*.$$

For all $n \geq 0$, we let $G_n := G(L_n/k)$, $\Gamma_n := G(L_n/L_0)$. Also, we let $G_\infty := G(L_\infty/k)$, $\Gamma_\infty := G(L_\infty/L_0)$. The results in the previous section show that we have the following commutative diagrams of abelian groups with exact rows and canonical vertical isomorphisms.

$$0 \longrightarrow U_{k_{\mathfrak{p}}}^{(1)}/U_{k_{\mathfrak{p}}}^{(n+1)} \longrightarrow (A/\mathfrak{fp}^{n+1})^{\times}/\mathbb{F}_q^{\times} \longrightarrow (A/\mathfrak{fp})^{\times}/\mathbb{F}_q^{\times} \longrightarrow 0$$

$$0 \longrightarrow \Gamma_n \longrightarrow G(L_n/H_{\mathfrak{c}}) \longrightarrow G(L_0/H_{\mathfrak{c}}) \longrightarrow 0$$

Consequently, for all $n \geq 0$, we obtain the following diagram of field extensions whose relative Galois groups are canonically isomorphic to the labels of the connecting line segments.

(2)

Further, we obtain topological group isomorphisms

$$\Gamma_{\infty} := G(L_{\infty}/L_0) \simeq G(L_{\infty}^*/L_0^*) \simeq \varprojlim_n \frac{U_{k_{\mathfrak{p}}}^{(1)}}{U_{k_{\mathfrak{p}}}^{(n+1)}} \simeq U_{k_{\mathfrak{p}}}^{(1)}.$$

Now, we recall the following structure theorem, due to Iwasawa. (See also [12, Satz II.5.7]).

Theorem 2.8 (Iwasawa [8]). *Let K be a local field of characteristic $p > 0$ and let $U_K^{(1)}$ denote its group of principal units. Then, there is an isomorphism of topological groups*

$$U_K^{(1)} \simeq \mathbb{Z}_p^{\aleph_0},$$

where the right side denotes a direct product of countably many copies of $(\mathbb{Z}_p, +)$, endowed with the product of the p–adic topologies.

As a consequence, we have an isomorphism of topological groups

(3) $\Gamma_{\infty} := G(L_{\infty}/L_0) \simeq \mathbb{Z}_p^{\aleph_0}.$

The following gives a description of the Iwasawa algebras relevant in our considerations below.

Proposition 2.9. *Let $(\mathcal{O}, \mathfrak{m}_{\mathcal{O}})$ be a local, compact $\mathbb{Z}_p$–algebra, which is $\mathfrak{m}_{\mathcal{O}}$–adically complete. If $\mathcal{G}$ is an abelian pro–p group, topologically isomorphic to $\mathbb{Z}_p^{\aleph_0}$, then the following hold.*

   (1) *There is an isomorphism of topological $\mathcal{O}$–algebras*

$$\mathcal{O}[[\mathcal{G}]] \simeq \mathcal{O}[[X_1, X_2, \dots]],$$

   where the left side is endowed with the profinite limit topology and the right side with the projective limit of the $(\mathfrak{m}_{\mathcal{O}}, X_1, \dots, X_n)$–adic topologies on each $\mathcal{O}[[X_1, \dots, X_n]]$, as $n \to \infty$.

   (2) *If $\mathcal{O}$ is an integral domain, then $\mathcal{O}[[\mathcal{G}]]$ is a local, integral domain.*

   (3) *If $\mathcal{O}$ is a PID, then $\mathcal{O}[[\mathcal{G}]]$ is a UFD and, therefore, normal.*

Proof. (Sketch.) (1) Use induction on n and the Weierstrass Preparation Theorem (see Thm. 2.1 in [11, Ch.5, §2]) to show that one has an isomorphism of topological $\mathcal{O}$–algebras

$$\mathcal{O}[[\mathbb{Z}_p^n]] \simeq \mathcal{O}[[X_1, \dots, X_n]].$$

Then, pass to a projective limit with respect to n to get the desired isomorphism.

   (2) This is Lemma 1 in [13]. Note that with the notations and definitions of loc.cit. we have $\mathcal{O}[[X_1, X_2, \dots]] = \mathcal{O}\{X\}_{\aleph_0}$, where X is a set of cardinality $\aleph_0$.

   (3) This is Theorem 1 in [13]. See the note above regarding the notations in loc.cit. $\quad\square$

Remark 2.10. Typical examples of $\mathbb{Z}_p$–algebras $\mathcal{O}$ as in the Proposition above are rings of integers $\mathcal{O}_F$ in finite extensions $F/\mathbb{Q}_p$ of $\mathbb{Q}_p$. Also, group rings $\mathcal{O}_F[P]$, where P is a finite, abelian p–group satisfy the hypotheses of part (1), but not parts (2)–(3) of the proposition above. Note that in the latter case the maximal ideal of $\mathcal{O}_F[P]$ is given by $\mathfrak{m}_{\mathcal{O}_F[P]} = (\mathfrak{m}_{\mathcal{O}_F}, I_P)$, where I_P is the augmentation ideal of $\mathcal{O}_F[P]$. Note that if P is a product of r cyclic groups of orders $p^{n_1}, \dots, p^{n_r}$, respectively, then we have isomorpisms of topological $\mathcal{O}_F$–algebras

$$\begin{aligned}
\mathcal{O}_F[P] &\simeq \mathcal{O}_F[X_1, \dots, X_r]/\left((X_1+1)^{p^{n_1}} - 1, \dots (X_r+1)^{p^{n_r}} - 1\right) \\
&\simeq O_F[[X_1, \dots, X_r]]/\left((X_1+1)^{p^{n_1}} - 1, \dots (X_r+1)^{p^{n_r}} - 1\right),
\end{aligned}$$

where the first isomorphism sends the generators of P to $\widehat{X_1 + 1}, \dots, \widehat{X_r + 1}$, respectively, and the second is a consequence of the Weierstrass preparation theorem cited above, applied inductively. Since the right-most algebra is clearly complete in its $(\mathfrak{m}_{\mathcal{O}_F}, X_1, \dots, X_r)$–adic topology, the left-most algebra is also complete in its $\mathfrak{m}_{\mathcal{O}_F[P]}$–adic topology.

    We end this section with a result on the decomposition groups $G_v(L_\infty/L_n)$ in the extension L_∞/L_n, for all primes $v|\mathfrak{f}$ and a fixed $n \geq 0$. This will be used in the proof of Proposition 3.22 below. To that end, fix $n \geq 0$ and for every prime $v|\mathfrak{f}$, let U_{S_v} be the group of S_v–units in $k^\times$, where $S_v := \{v, \infty\}$. We remind the reader that these are the elements of $k^\times$ whose divisor is supported at S_v. Consequently, we have a group isomorphism

$$U_{S_v} \simeq \mathbb{F}_q^\times \times \mathbb{Z}.$$

Further we let $U_{S_v}^{(n+1)} := \{x \in U_{S_v} \mid x \equiv 1 \mod (\frac{\mathfrak{f}}{v^{\mathrm{ord}_v(\mathfrak{f})}} \cdot \mathfrak{p}^{n+1})\}$. This is a subgroup of finite index in U_{S_v} which is torsion free. Therefore, it is infinite cyclic

$$U_{S_v}^{(n+1)} = x_v^{\mathbb{Z}},$$

generated by some $x_v \in k^\times$, which obviously satisfies the following

$$(4) \qquad \mathrm{div}(x_v) = \mathrm{ord}_v(x_v) \cdot v + \mathrm{ord}_\infty(x_v) \cdot \infty, \quad \mathrm{ord}_v(x_v) = -(d_\infty/d_v) \cdot \mathrm{ord}_\infty(x_v) \neq 0.$$

In what follows, we let $U(\mathfrak{fp}^\infty) := \bigcap_n U(\mathfrak{fp}^n)$ and let $i_v : k_v^\times \to J_k/k^\times U(\mathfrak{fp}^\infty)$ be the standard morphism (sending $x \in k_v^\times$ into the class of the idèle having x in the v–component and 1 everywhere else), for all primes v of k. Now, we consider the topological group isomorphism

$$\rho_\mathfrak{p}^{(n)} : U_{k_\mathfrak{p}}^{(n+1)} \simeq G(L_\infty/L_n)$$

obtained by composing the Artin reciprocity isomorphism $\rho \colon J_k/k^\times U(\mathfrak{fp}^\infty) \simeq G(L_\infty/k)$ with the standard embedding $U_{k_\mathfrak{p}}^{(n+1)} \subseteq k_\mathfrak{p}^\times \xrightarrow{i_\mathfrak{p}} J_k/k^\times U(\mathfrak{fp}^\infty)$. Note that $i_\mathfrak{p}$ restricted to $U_{k_\mathfrak{p}}^{(n+1)}$ is indeed injective, for all $n \geq 0$.

Proposition 2.11. *With notations as above, the following hold.*

(1) *For all primes $v|\mathfrak{f}$, if we let $x_v^{\mathbb{Z}_p}$ denote the cyclic $\mathbb{Z}_p$–submodule of $U_{k_\mathfrak{p}}^{(n+1)}$ generated by x_v, then $\rho_\mathfrak{p}^{(n)}$ gives an isomorphism of topological groups*

$$\rho_\mathfrak{p}^{(n)} : x_v^{\mathbb{Z}_p} \simeq G_v(L_\infty/L_n).$$

(2) *Let $G_\mathfrak{f}(L_\infty/L_n)$ be the subgroup of $G(L_\infty/L_n)$ generated by $G_v(L_\infty/L_n)$, for all $v|\mathfrak{f}$. Then, if we let $f := \mathrm{card}\{v \mid v|\mathfrak{f}\}$, we have topological group isomorphisms*

$$G_\mathfrak{f}(L_\infty/L_n) \simeq \prod_{v|\mathfrak{f}} G_v(L_\infty/L_n) \simeq \mathbb{Z}_p^f$$

Proof. (1) A well–known class–field theoretical fact gives an equality of groups

$$G_v(L_\infty/L_n) = \overline{\rho(i_v(k_v^\times)) \cap \rho(i_\mathfrak{p}(U_{k_\mathfrak{p}}^{(n+1)}))},$$

where $\overline{X}$ denotes the pro-p completion (topological closure) of the subgroup X inside the pro-p group $G(L_\infty/L_n)$. However, it is easily seen that we have

$$\rho(i_v(k_v^\times)) \cap \rho(i_\mathfrak{p}(U_{k_\mathfrak{p}}^{(n+1)})) = \rho_\mathfrak{p}^{(n)}(x_v^{\mathbb{Z}}),$$

which, after taking the pro–p completion of both sides, concludes the proof of part (1).

(2) According to part (1), it suffices to show that the elements $\{x_v \mid v|\mathfrak{f}\}$ are $\mathbb{Z}_p$–linearly independent in $U_{k_\mathfrak{p}}^{(n+1)}$. However, since their divisors are clearly $\mathbb{Z}$–linearly independent (see (4) above), these elements are $\mathbb{Z}$–linearly independent in $k^\times$. Now, the function field (strong) analogue of Leopoldt's Conjecture, proved in [10], implies that the elements in question are $\mathbb{Z}_p$–linearly independent in $U_{k_\mathfrak{p}}^{(n+1)}$, as desired. $\qquad\square$

2.3. **The basic example: The Carlitz module.** We briefly describe the special situation which arises in the case of the Carlitz cyclotomic extension of a rational function field (see e.g. [1, Sec. 2]).

Let $k = \mathbb{F}_q(\theta)$ be the rational function field over $\mathbb{F}_q$ and let v_∞ correspond to the valuation on k of uniformizer $1/\theta$. Then $A = \mathbb{F}_q[\theta]$. Furthermore, $h_k = 1$, $d_\infty = 1$, and $H_\mathfrak{e}^* = H_\mathfrak{e} = k$. We consider the Carlitz module

$$\mathcal{C} \colon A \longrightarrow k\{\tau\}, \quad \theta \mapsto \mathcal{C}(\theta) = \theta\tau^0 + \tau^1,$$

which is sgn–normalized with respect to the unique sign function satisfying $\mathrm{sgn}(1/\theta) = 1$. All data in the following refers to $\rho = \mathcal{C}$.

For each $\mathfrak{m} \neq \mathfrak{e}$ we have

$$\mathrm{G}(H_{\mathfrak{m}}^*/k) \simeq (A/\mathfrak{m})^{\times}, \quad \mathrm{G}(H_{\mathfrak{m}}/k) \simeq (A/\mathfrak{m})^{\times}/\mathbb{F}_q^{\times}, \quad \mathrm{G}(H_{\mathfrak{m}}^*/H_{\mathfrak{m}}) \simeq \mathbb{F}_q^{\times} = \mathbb{F}_{\infty}^{\times},$$

and the subgroup $\mathbb{F}_q^{\times} \hookrightarrow (A/\mathfrak{m})^{\times}$ identifies with the decomposition subgroup at ∞ which also equals the ramification subgroup at v_{∞} for the extension $H_{\mathfrak{m}}^*/k$.

We fix a prime $\mathfrak{p}$ of A of degree $d = d_{\mathfrak{p}}$ and consider the fields $L_n^* := H_{\mathfrak{p}^{n+1}}^*$ for $n \geq 0$. Then

$$(5) \qquad G_n^* := \mathrm{G}(L_n^*/k) = \Delta^* \times \Gamma_n \simeq \left(A/\mathfrak{p}^{n+1}\right)^{\times} \simeq (A/\mathfrak{p})^{\times} \times \frac{U_{k_{\mathfrak{p}}}^{(1)}}{U_{k_{\mathfrak{p}}}^{(n+1)}},$$

where $\Delta^* \simeq \mathrm{G}(L_0^*/k) \simeq (A/\mathfrak{p})^{\times}$ is cyclic of order $q^d - 1$ and $\Gamma_n \simeq \mathrm{G}(L_n^*/L_0^*) \simeq U_{k_{\mathfrak{p}}}^{(1)}/U_{k_{\mathfrak{p}}}^{(n+1)}$ is the p-Sylow subgroup of G_n^*.

The extension L_n^*/k is unramified outside $\{v_{\infty}, \mathfrak{p}\}$, totally ramified at $\mathfrak{p}$ and tamely ramified of ramification degree $(q - 1)$ at v_{∞}. More precisely, the decomposition field at v_{∞} is $L_n := H_{\mathfrak{p}^{n+1}}$ and L_n^*/L_n is totally ramified of degree $(q - 1)$.

Hence, the extension L_{∞}^*/k is also unramified outside $\{v_{\infty}, \mathfrak{p}\}$, totally ramified at $\mathfrak{p}$ and tamely ramified of degree $(q - 1)$ at v_{∞}. More precisely, the decomposition field at v_{∞} is $L_{\infty} := \cup_n H_{\mathfrak{p}^n}$ and L_{∞}^*/L_{∞} is totally ramified at v_{∞} of degree $(q - 1)$. We have

$$G_{\infty}^* := \mathrm{G}(L_{\infty}^*/k) = \Delta^* \times \Gamma_{\infty}, \quad \Gamma_{\infty} \simeq U_{k_{\mathfrak{p}}}^{(1)}.$$

Here the isomorphism $\Gamma_{\infty} \simeq U_{k_{\mathfrak{p}}}^{(1)}$ is induced by the $\mathfrak{p}$-cyclotomic character

$$\kappa \colon G_{\infty} \longrightarrow U_{k_{\mathfrak{p}}},$$

which is defined as follows. We write $A_{\mathfrak{p}}$ for the completion of A at $\mathfrak{p}$, so that $A_{\mathfrak{p}}$ identifies with the valuation ring of $k_{\mathfrak{p}}$, in particular, $A_{\mathfrak{p}}^{\times} = U_{k_{\mathfrak{p}}}$. Then, ϕ can be uniquely extended to a formal Drinfeld module (see [16])

$$\widehat{\mathcal{C}} \colon A_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}}\{\{\tau\}\}.$$

Then, for any $\sigma \in G_{\infty}$ the value $\kappa(\sigma)$ is determined by the equality

$$\sigma(\epsilon) = \widehat{\mathcal{C}}_{\kappa(\sigma)}(\epsilon) \text{ for all } \epsilon \in \mathcal{C}[\mathfrak{p}^{\infty}].$$

Finally, we note that

$$G_{\infty} := \mathrm{G}(L_{\infty}/k) = \Delta \times \Gamma_{\infty},$$

where $\Delta := \Delta^*/\mathbb{F}_q^{\times} \simeq (A/\mathfrak{p})^{\times}/\mathbb{F}_q^{\times}$.

3. EQUIVARIANT MAIN CONJECTURES IN POSITIVE CHARACTERISTIC

3.1. **Review of the work of Greither and Popescu.** In what follows, if G is a finite, abelian group and F is a field of characteristic 0, we denote by $\widehat{G}(F)$ the set of equivalence classes of the $\overline{F}$–valued characters χ of G, with respect to the equivalence relation $\chi \sim \chi'$ if there exists $\sigma \in G(\overline{F}/F)$, such that $\chi' = \sigma \circ \chi$.

If R is a commutative ring and M is a finitely presented R–module, we let $\mathrm{Fitt}_R(M)$ denote the 0–th Fitting ideal of M. For the definitions and relevant properties of Fitting ideals needed in this context, the reader may consult [5].

We let K/k denote an abelian extension of characteristic p global fields, of Galois group G. We assume that $\mathbb{F}_q$ is the exact field of constants of k (but not necessarily of K). Let $X \longrightarrow Y$ be the corresponding G-Galois cover of smooth projective curves defined over $\mathbb{F}_q$. Let S and Σ be two finite, non-empty, disjoint sets of closed points of Y, such that S contains the set S_{ram} of points which ramify in X. We let $\overline{\mathbb{F}_q}$ denote the algebraic closure of $\mathbb{F}_q$ and set $\bar{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, $\bar{Y} := Y \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. Also, $\bar{S}$ and $\bar{\Sigma}$ denote the set of points on $\bar{X}$ sitting above points of S and Σ, respectively.

For every unramified closed point v on Y we denote by G_v and σ_v the decomposition group and the Frobenius automorphism associated to v. As before, we write d_v for the residual degree over $\mathbb{F}_q$ and we let $Nv := q^{d_v} = |\mathbb{F}_{q^{d_v}}|$ denote the cardinality of the residue field associated to v.

To the set of data $(K/k, \mathbb{F}_q, S, \Sigma)$, one can associate a polynomial equivariant L-function

$$(6) \qquad \Theta_{S,\Sigma}(u) := \prod_{v \in \Sigma} \left(1 - \sigma_v^{-1} \cdot (qu)^{d_v}\right) \cdot \prod_{v \notin S} \left(1 - \sigma_v^{-1} \cdot u^{d_v}\right)^{-1}.$$

The infinite product on the right is taken over all closed points in Y which are not in S. This product converges in $\mathbb{Z}[G][[u]]$ and in fact it converges to an element in the polynomial ring $\mathbb{Z}[G][u]$. We recall the link between $\Theta_{S,\Sigma}(u)$ and classical Artin L-functions. For every complex valued irreducible character χ of G we let $L_{S,\Sigma}$ denote the (S, Σ)-modified Artin L-function associated to χ. This is the unique holomorphic function of the complex variable s satisfying the equality

$$(7) \qquad L_{S,\Sigma}(\chi, s) = \prod_{v \in \Sigma} \left(1 - \chi(\sigma_v)Nv^{1-s}\right) \cdot \prod_{v \notin S} \left(1 - \chi(\sigma_v)(Nv)^{-s}\right)^{-1}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Then, for all $s \in \mathbb{C}$,

$$(8) \qquad \Theta_{S,\Sigma}(q^{-s}) = \sum_{\chi \in \widehat{G}(\mathbb{C})} L_{S,\Sigma}(\chi, s)e_{\chi^{-1}},$$

where $e_\chi := 1/|G| \sum_{g \in G} \chi(g)g^{-1} \in \mathbb{C}[G]$ denotes the idempotent corresponding to χ.

We denote by $M_{\bar{S},\bar{\Sigma}}$ the Picard 1-motive associated to the set of data $(\bar{X}, \overline{\mathbb{F}_q}, \bar{S}, \bar{\Sigma})$, see [5, Def. 2.3] for the definition. For a prime number ℓ we consider the ℓ-adic Tate module (or ℓ-adic realization) $T_\ell(M_{\bar{S},\bar{\Sigma}})$, see [5, Def. 2.6], endowed with the usual $\mathbb{Z}_\ell[[G \times \Gamma]]$-module structure, where $\Gamma := G(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. Recall that Γ is isomorphic to the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ and has a natural topological generator γ given by the q-power arithmetic Frobenius automorphism.

The main result of Section 4 of [5] is the following.

Theorem 3.1 (Greither–Popescu)**.** *The following hold for all prime numbers ℓ.*

(1) *The $\mathbb{Z}_\ell[G]$-module $T_\ell(M_{\bar{S},\bar{\Sigma}})$ is projective.*
(2) *We have an equality of $\mathbb{Z}_\ell[[G \times \Gamma]]$-ideals*

$$\left(\Theta_{S,\Sigma}(\gamma^{-1})\right) = \mathrm{Fitt}_{\mathbb{Z}_\ell[[G\times\Gamma]]}(T_\ell(M_{\bar{S},\bar{\Sigma}})).$$

Remark 3.2. By [5, Rem. 2.7] we have $T_p(M_{\bar{S},\bar{\Sigma}}) = T_p(M_{\bar{S},\emptyset})$. This is in accordance with the fact that the product of Euler factors

$$\prod_{v\in\Sigma}\left(1 - \sigma_v^{-1} \cdot (q\gamma^{-1})^{d_v}\right)$$

is a unit in $\mathbb{Z}_\ell[[G \times \Gamma]]$. Indeed, from

$$\sum_{n=0}^{N-1}\left(\sigma_v^{-1}(qu)^{d_v}\right)^n = \frac{1 - \left(\sigma_v^{-1}(qu)^{d_v}\right)^N}{1 - \left(\sigma_v^{-1}(qu)^{d_v}\right)}$$

and $\lim_{N\to\infty}\left(1 - \left(\sigma_v^{-1}(qu)^{d_v}\right)^N\right) = 1$ in $\mathbb{Z}_p[G][[u]]$, we see that

$$\frac{1}{1 - \left(\sigma_v^{-1}(qu)^{d_v}\right)} = \sum_{n=0}^\infty\left(\sigma_v^{-1}(qu)^{d_v}\right)^n \in \mathbb{Z}_p[G][[u]].$$

3.2. **The main results (Geometric Equivariant Main Conjectures).** We fix a prime ideal $\mathfrak{p}$ and an integral ideal $\mathfrak{f}$ of A, such that $\mathfrak{p} \nmid \mathfrak{f}$. We will consider the tower of fields $H_{\mathfrak{fp}^{n+1}}/k$, for $n \geq 0$. (See §2.2 and field diagram (2).) The definition of the real ray–class fields $H_{\mathfrak{fp}^n}$ implies that we have

$$H_{\mathfrak{fp}^n} \cap \bar{\mathbb{F}}_q = \mathbb{F}_{q^{d_\infty}}, \qquad \bar{\mathbb{F}}_q H_{\mathfrak{f}} \cap H_{\mathfrak{fp}^n} = H_{\mathfrak{f}},$$

for all $n \geq 0$. Consequently, we have the following perfect field diagram.

(9)

$$\mathcal{L}_n := \bar{\mathbb{F}}_q H_{\mathfrak{f}\mathfrak{p}^{n+1}}$$

$$L_n := H_{\mathfrak{f}\mathfrak{p}^{n+1}} \qquad \mathcal{L}_0 := \bar{\mathbb{F}}_q H_{\mathfrak{f}\mathfrak{p}}$$

$$\mathcal{K} := \bar{\mathbb{F}}_q k$$

$$G_n$$

$$L_0 := H_{\mathfrak{f}\mathfrak{p}} \qquad \kappa := \bar{\mathbb{F}}_q$$

$$G_0$$

$$E := L_0 \cap \mathcal{K}$$

$$k \qquad \mathbb{F}_{q^{d_\infty}} \qquad \Gamma$$

$$\mathbb{F}_q$$

As in Subsection 3.1 we let S and Σ be two finite, non-empty, disjoint sets of closed points of the smooth, projective curve Y corresponding to k, such that S contains the set S_{ram} of points which ramify in $H_{\mathfrak{f}\mathfrak{p}^{n+1}}$. Note that this condition does not depend on n.

We write $\Theta_{S,\Sigma}^{(n)}(u) \in \mathbb{Z}[G_n][u]$ for the equivariant L-function attached to $(L_n/k, \mathbb{F}_q, S, \Sigma)$ in (6). We let $L_\infty := \cup_n L_n$ and $G_\infty := \mathrm{Gal}(L_\infty/k)$. The next lemma shows that the following is well defined.

$$\Theta_{S,\Sigma}^{(\infty)}(u) := \varprojlim_n \Theta_{S,\Sigma}^{(n)}(u) \in \mathbb{Z}_p[[G_\infty]][[u]].$$

Lemma 3.3. *Let L/k be a finite abelian extension with Galois group $G := \mathrm{G}(L/k)$. Let K/k be a subextension with $H := \mathrm{G}(L/K)$. We write*

$$\Theta_{S,\Sigma,L/k}(u) \in \mathbb{Z}_p[G][u], \quad \Theta_{S,\Sigma,K/k}(u) \in \mathbb{Z}_p[G/H][u]$$

for the equivariant L-functions attached to the data $(L/k, \mathbb{F}_q, S, \Sigma)$ and $(K/k, \mathbb{F}_q, S, \Sigma)$, respectively. Then the canonical map $\mathbb{Z}_p[G][u] \longrightarrow \mathbb{Z}_p[G/H][u]$ sends $\Theta_{S,\Sigma,L/k}(u)$ to $\Theta_{S,\Sigma,K/k}(u)$.

Proof. We write π for the canonical map $G \longrightarrow G/H$ and also for any map which is naturally induced by π. It is straightforward to verify that for any character $\chi \in \hat{G}$ one has

$$\pi(e_\chi) = \begin{cases} e_\psi, & \text{if } \chi|_H = 1 \text{ and } \chi = \mathrm{inf}_{G/H}^G(\psi), \\ 0, & \text{if } \chi|_H \neq 1. \end{cases}$$

Hence, by the inflation invariance of (S, Σ)-modified Artin L-functions, we obtain

$$\pi \left(\sum_{\chi \in \hat{G}} L_{S,\Sigma}(\chi, s) e_{\chi^{-1}} \right) = \sum_{\psi \in \widehat{G/H}} L_{S,\Sigma}(\psi, s) e_{\psi^{-1}}.$$

It follows that $(\pi(\Theta_{S,\Sigma,L/k}))(q^{-s}) = \Theta_{S,\Sigma,K/k}(q^{-s})$ for all $s \in \mathbb{C}$, and hence we also have $\pi(\Theta_{S,\Sigma,L/k}(u)) = \Theta_{S,\Sigma,K/k}(u)$ by (8). $\qquad\square$

We let $X_n \longrightarrow Y$ denote the G_n-Galois cover of smooth, projective curves defined over $\mathbb{F}_q$ corresponding to L_n/k. We write $\overline{X}_n := X_n \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, $\overline{Y} := Y \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, and also $\bar{S}_n$ for the set of points of $\bar{X}_n$ above points of S. We let $M_S^{(n)}$ be the Picard 1-motive associated with the set of data $(\bar{X}_n, \overline{\mathbb{F}_q}, \bar{S}_n, \emptyset)$ and write $T_p^{(n)} := T_p(M_S^{(n)})$ for the p-adic Tate module of $M_S^{(n)}$. Then, $T_p^{(n)}$ is endowed with a natural structure of $\mathbb{Z}_p[G_n][[\Gamma]]$–module. (See [5, Sec. 3].)

Now, since $\mathbb{F}_{q^{d_\infty}}$ is the exact field of constants of L_n, $\overline{X}_n$ is going to have d_∞ connected components, all isomorphic to $\overline{X}_n^c := X_n \times_{\mathbb{F}_{q^{d_\infty}}} \overline{\mathbb{F}_q}$. We let $T_{p,c}^{(n)} := T_{p,c}(M_S^{(0)}) := T_p(M_S^{(n),c})$ denote the p-adic Tate module of the 1–motive $M_S^{(n),c}$ associated to the data $(\bar{X}_n^c, \overline{\mathbb{F}_q}, \bar{S}_n \cap \overline{X}_n^c, \emptyset)$.

For $m \geq n \geq 0$ we write $N_{m/n} \colon L_m^\times \longrightarrow L_n^\times$ for the field theoretic norm map and also view $N_{m/n} = \sum_{g \in \mathrm{G}(L_m/L_n)} g$ as an element of $\mathbb{Z}_p[G_m]$. Galois restriction gives isomorphisms

$$\mathrm{G}(\mathcal{L}_m/\mathcal{L}_n) \simeq \mathrm{G}(L_m/L_n).$$

By the results of [5, Sec. 3] we have a natural $\mathbb{Z}_p[G_{n+1}]$–equivariant, injective morphism $T_p^{(n)} \hookrightarrow T_p^{(n+1)}$. We may also consider the norm map

$$N_{n+1/n} \colon T_p^{(n+1)} \longrightarrow \left(T_p^{(n+1)} \right)^{\mathrm{G}(L_{n+1}/L_n)}.$$

By [5, Th. 3.10] this norm map is surjective and, moreover, it is an almost formal consequence of [5, Th. 3.1] to show that, under the canonical injective morphism $T_p^{(n)} \hookrightarrow T_p^{(n+1)}$, we can identify the following $\mathbb{Z}_p[G_n]$–modules

$$(10) \qquad \left(T_p^{(n+1)} \right)^{\mathrm{G}(L_{n+1}/L_n)} = T_p^{(n)}.$$

Indeed, we recall the diagram of fields above and set for $n \geq 0$

$$H_n := \mathrm{G}(L_n/E).$$

Then we have natural isomorphisms of $\mathbb{Z}_p[G_n]$–modules (see [5, proof of Th. 3.10])

$$T_p^{(n)} \simeq T_{p,c}^{(n)} \otimes_{\mathbb{Z}_p[H_n]} \mathbb{Z}_p[G_n].$$

Therefore, we have the following

$$
\begin{aligned}
\left(T_p^{(n+1)}\right)^{\mathrm{G}(L_{n+1}/L_n)} &\simeq \left(T_{p,c}^{(n+1)} \otimes_{\mathbb{Z}_p[H_{n+1}]} \mathbb{Z}_p[G_{n+1}]\right)^{\mathrm{G}(L_{n+1}/L_n)} \\
&\overset{(*)}{=} \left(T_{p,c}^{(n+1)}\right)^{\mathrm{G}(L_{n+1}/L_n)} \otimes_{\mathbb{Z}_p[H_n]} \mathbb{Z}_p[G_n] \\
&\overset{(**)}{=} \left(T_{p,c}^{(n)}\right) \otimes_{\mathbb{Z}_p[H_n]} \mathbb{Z}_p[G_n] \\
&\simeq T_p^{(n)},
\end{aligned}
$$

where $(*)$ is easy to verify using the fact that $T_{p,c}^{(n+1)}$ is $\mathbb{Z}_p[H_{n+1}]$-projective by Theorem 3.1 and $(**)$ is the result of [5, Th. 3.1].

Definition 3.4. We define the Iwasawa–type algebras

$$
\Lambda_n := \mathbb{Z}_p[[G_n \times \Gamma]], \qquad \Lambda := \varprojlim_n \Lambda_n = \mathbb{Z}_p[[G_\infty \times \Gamma]],
$$

and consider the Λ–module defined by

$$
T_p(M_S^{(\infty)}) := \varprojlim_n T_p(M_S^{(n)}),
$$

where the projective limit is taken with respect to norm maps.

The rest of this section is devoted to the proof of Theorem 1.2 which we recall for the reader's convenience.

Theorem 3.5 (EMC-I). *Let S and Σ be as above. Then the Λ–module $T_p(M_S^{(\infty)})$ is finitely generated and torsion and the following hold.*

(1) $\mathrm{pd}_\Lambda \left(T_p(M_S^{(\infty)})\right) = 1.$

(2) $\mathrm{Fitt}_\Lambda \left(T_p(M_S^{(\infty)})\right) = \Theta_{S,\Sigma}^{(\infty)}(\gamma^{-1}) \cdot \Lambda.$

The strategy of proof is as follows. First, we obtain a finitely generated, Λ–projective resolution of length 1 for $T_p(M_S^{(\infty)})$, as a projective limit of certain Λ_n–projective resolutions of length 1 for $T_p(M_S^{(n)})$, for $n \geq 0$, essentially constructed in [5]. This implies that $T_p(M_S^{(\infty)})$ is finitely generated and $\mathrm{pd}_\Lambda \leq 1$. Then, we use this construction further to show that

$$
(11) \qquad \mathrm{Fitt}_\Lambda \left(T_p(M_S^{(\infty)})\right) = \varprojlim_n \mathrm{Fitt}_{\Lambda_n} \left(T_p(M_S^{(n)})\right).
$$

Next, we show that $\Theta_{S,\Sigma}^{(n)}(\gamma^{-1})$ is a non-zero divisor in Λ_n, for all $n \geq 0$ and $\Theta_{S,\Sigma}^{(\infty)}(\gamma^{-1})$ is a non–zero divisor in Λ. (See Corollary 3.12 below.) When combined with Theorem 3.1(2), equality (11) and Lemma 3.10 below, this leads to the equalities

$$
\mathrm{Fitt}_\Lambda \left(T_p(M_S^{(\infty)})\right) = \varprojlim_n \left(\Theta_{S,\Sigma}^{(n)}(\gamma^{-1}) \cdot \Lambda_n\right) = \Theta_{S,\Sigma}^{(\infty)}(\gamma^{-1}) \cdot \Lambda.
$$

Now, the fact that $T_p(M_S^{(\infty)})$ is Λ–torsion and of projective dimension exactly equal to 1 follows from the following elementary result.

Lemma 3.6. *Let R be a commutative ring and X a finitely generated R–module. Assume that $\mathrm{Fitt}_R(X)$ contains a non–zero divisor $f \in R$. Then X is a torsion R–module. Consequently, if X is non–zero, then X cannot be a submodule of a free R–module and therefore it cannot be R–projective.*

Proof. From the well–known inclusion $\mathrm{Fitt}_R(X) \subseteq \mathrm{Ann}_R(X)$, we conclude that $f \cdot X = 0$, which concludes the proof of the Lemma. $\qquad\square$

From now on, we let P_n denote the Sylow p–subgroup of G_n and Δ_n its complement, so that $G_n = P_n \times \Delta_n$, for all $n \geq 0$. Note that since $\Gamma_n = G(L_n/L_0)$ is a p–group, $\Delta := \Delta_n$ does not depend on n. Consequently, for all $n \geq 0$, we have

(12) $$G_n \simeq P_n \times \Delta, \qquad P_n/\Gamma_n \simeq P_0.$$

Therefore, for all $n \geq 0$, we have an isomorphism of $\mathbb{Z}_p$–algebras

$$\mathbb{Z}_p[G_n] \simeq \bigoplus_{\chi \in \hat{\Delta}(\mathbb{Q}_p)} \mathbb{Z}_p(\chi)[P_n],$$

given by the usual direct sum of χ–evaluation maps for $\chi \in \hat{\Delta}$. Consequently, any $\mathbb{Z}_p[G_n]$–module X splits naturally into a direct sum

$$X = \bigoplus_{\chi \in \hat{\Delta}(\mathbb{Q}_p)} X^\chi, \qquad \text{where } X^\chi \simeq X \otimes_{\mathbb{Z}_p[G_n]} \mathbb{Z}_p(\chi)[P_n].$$

Since P_n is an abelian p–group, the rings $\mathbb{Z}_p(\chi)[P_n]$ are local rings, for all n and χ as above. Further, since projective modules over local rings are free, Theorem 3.1 and [5, Rem. 2.7] imply that we have isomorphisms of $\mathbb{Z}_p(\chi)[P_n]$–module

(13) $$T_p\left(M_S^{(n)}\right)^\chi \simeq (\mathbb{Z}_p(\chi)[P_n])^{m_\chi^{(n)}},$$

with integers $m_\chi^{(n)} \geq 0$, for all χ and n as above.

Lemma 3.7. *The non-negative integers $m_\chi^{(n)}$ do not depend on n.*

Proof. Note that since $T_p(M_S^{(n)})$ is $\mathbb{Z}_p[G_n]$–projective, taking $G(L_n/L_0)$ fixed points commutes with taking χ-parts. Hence (13) combined with (10) implies $m_\chi^{(n)} = m_\chi^{(0)}$, for all characters χ and all $n \geq 0$. $\qquad\square$

Definition 3.8. We let $m_\chi := m_\chi^{(n)} = \mathrm{rank}_{\mathbb{Z}_p(\chi)[P_n]} T_p(M_S^{(n)})^\chi$, for all χ and n as above.

In order to simplify notations, for every $\chi \in \hat{\Delta}$ and all $n \geq 0$, we write

$$R_n := \mathbb{Z}_p[G_n], \quad R_n^\chi := \mathbb{Z}_p(\chi)[P_n], \quad T_n = T_p\left(M_S^{(n)}\right), \quad T_n^\chi = T_p\left(M_S^{(n)}\right)^\chi.$$

We let $P_\infty := \varprojlim_n P_n$, observe that $G_\infty = P_\infty \times \Delta$, and set

$$R_\infty := \mathbb{Z}_p[[G_\infty]], \quad R_\infty^\chi := \mathbb{Z}_p(\chi)[[P_\infty]], \quad T_\infty := T_p(M_S^{(\infty)}), \quad T_\infty^\chi := T_p(M_S^{(\infty)})^\chi.$$

Further, we let $\Lambda_n^\chi := R_n^\chi[[\Gamma]] = \mathbb{Z}_p(\chi)[[P_n \times \Gamma]]$ and $\Lambda^\chi := R_\infty^\chi[[\Gamma]] = \mathbb{Z}_p(\chi)[[P_\infty \times \Gamma]]$. Since

$$\Lambda_n \simeq \bigoplus_{\chi \in \widehat{\Delta}(\mathbb{Q}_p)} \Lambda_n^\chi, \quad \Lambda \simeq \bigoplus_{\chi \in \widehat{\Delta}(\mathbb{Q}_p)} \Lambda^\chi,$$

via the usual character–evaluation maps, the Λ_n–modules Λ_n^χ and the Λ–modules Λ^χ are projective and cyclic, for all characters χ as above.

Now, we fix a character χ as above and, for a given $n \geq 0$, we fix an R_n^χ-basis of T_n^χ:

$$x_1^{(n)}, \ldots, x_{m_\chi}^{(n)}.$$

We let $A_\gamma^{(n),\chi} \in \mathrm{GL}_{m_\chi}(R_n^\chi)$ be the matrix associated to the action of γ on T_n^χ with respect to the fixed basis. Let $\Phi_\gamma^{(n),\chi}$ be the $R_n^\chi[[\Gamma]]$-linear endomorphism of $R_n^\chi[[\Gamma]]^{m_\chi}$ of matrix

$$1 - \gamma^{-1} A_\gamma^{(n),\chi} \in M_{m_\chi}(R_n^\chi[[\Gamma]])$$

with respect to the canonical $R_n^\chi[[\Gamma]]$-basis $e_1^{(n)}, \ldots, e_{m_\chi}^{(n)}$ of $R_n^\chi[[\Gamma]]^{m_\chi}$. By the proof of [5, Prop. 4.1], in particular (6) of loc.cit., combined with Corollary 3.12(2) below, we have an exact sequence of $R_n^\chi[[\Gamma]]$–modules

(14) $$0 \longrightarrow R_n^\chi[[\Gamma]]^{m_\chi} \xrightarrow{\Phi_\gamma^{(n),\chi}} R_n^\chi[[\Gamma]]^{m_\chi} \xrightarrow{\pi_n^\chi} T_n^\chi \longrightarrow 0,$$

where π_n^χ is defined by $\pi_n^\chi(e_i^{(n)}) = x_i^{(n)}$.

Next, we show that, given an R_n^χ–basis $x_1^{(n)}, \ldots, x_{m_\chi}^{(n)}$ for T_n^χ, we can choose an R_{n+1}^χ-basis $x_1^{(n+1)}, \ldots, x_{m_\chi}^{(n+1)}$ of T_{n+1}^χ, such that we have a commutative diagram

(15)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_n^\chi[[\Gamma]]^{m_\chi} & \xrightarrow{\Phi_\gamma^{(n),\chi}} & R_n^\chi[[\Gamma]]^{m_\chi} & \xrightarrow{\pi_n^\chi} & T_n^\chi & \longrightarrow & 0 \\
& & \varphi_{n+1/n} \big\uparrow & & \varphi_{n+1/n} \big\uparrow & & N_{n+1/n} \big\uparrow & & \\
0 & \longrightarrow & R_{n+1}^\chi[[\Gamma]]^{m_\chi} & \xrightarrow{\Phi_\gamma^{(n+1),\chi}} & R_{n+1}^\chi[[\Gamma]]^{m_\chi} & \xrightarrow{\pi_{n+1}^\chi} & T_{n+1}^\chi & \longrightarrow & 0.
\end{array}$$

Here the vertical maps on the left and in middle are defined by $e_i^{(n+1)} \mapsto e_i^{(n)}$ and the canonical (componentwise) projections $R_{n+1}^\chi \to R_n^\chi$.

To that end, we start with an arbitrary R_{n+1}^χ-basis

$$y_1^{(n+1)}, \ldots, y_{m_\chi}^{(n+1)}$$

of T_{n+1}^χ and show how to modify it so that diagram (15) commutes. Since the module T_{n+1} is G_{n+1}–cohomologically trivial (see [5, Th. 3.10]), by (10) above we have

$$N_{n+1/n}\left(T_{n+1}^\chi\right) = \left(T_{n+1}^\chi\right)^{\mathrm{G}(L_{n+1}/L_n)} = T_n^\chi$$

and therefore

$$\left\{ N_{n+1/n}\left(y_1^{(n+1)}\right), \ldots, N_{n+1/n}\left(y_{m_\chi}^{(n+1)}\right) \right\}$$

is an R_n^χ-basis of T_n^χ. Let $U_n \in \mathrm{GL}_{m_\chi}(R_n^\chi)$ denote the matrix such that

$$
\begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_{m_\chi}^{(n)} \end{pmatrix} = U_n \begin{pmatrix} N_{n+1/n}\left(y_1^{(n+1)}\right) \\ \vdots \\ N_{n+1/n}\left(y_{m_\chi}^{(n+1)}\right) \end{pmatrix}.
$$

Let $U_{n+1} \in M_{m_\chi}(R_{n+1}^\chi)$ be such that $\varphi_{n+1/n}(U_{n+1}) = U_n$. Actually, by the next lemma, U_{n+1} is an invertible matrix.

Lemma 3.9. *Let $\varphi \colon S \longrightarrow R$ be a morphism of commutative local rings, i.e. $\varphi(\mathfrak{m}_S) \subseteq \mathfrak{m}_R$, where $\mathfrak{m}_S$ and $\mathfrak{m}_R$ are the corresponding maximal ideals. Let $U \in M_m(R)$, $V \in M_m(S)$ be matrices such that $\varphi(V) = U$. Then:*

$$
U \in \mathrm{GL}_m(R) \iff V \in \mathrm{GL}_m(S).
$$

Proof. If $VW = 1$ with $W \in M_m(S)$, then $1 = \varphi(VW) = \varphi(V) \cdot \varphi(W) = U \cdot \varphi(W)$, i.e., $U^{-1} = \varphi(W)$. Conversely suppose that $V \notin \mathrm{GL}_m(S)$. Then $\det_S(V) \in \mathfrak{m}_S$, and hence

$$
\det_R(U) = \det_R(\varphi(V)) = \varphi(\det_R(V)) \in \mathfrak{m}_R,
$$

contradicting $U \in \mathrm{GL}_m(R)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Now, since U_{n+1} is invertible, $\{x_1^{(n+1)}, \ldots, x_{m_\chi}^{(n+1)}\}$ defined by

$$
\begin{pmatrix} x_1^{(n+1)} \\ \vdots \\ x_{m_\chi}^{(n+1)} \end{pmatrix} = U_{n+1} \begin{pmatrix} y_1^{(n+1)} \\ \vdots \\ y_{m_\chi}^{(n+1)} \end{pmatrix}.
$$

is an R_{n+1}^χ–basis of T_{n+1}^χ. Then the right hand square of (15) commutes because
(16)

$$
\begin{pmatrix} N_{n+1/n}x_1^{(n+1)} \\ \vdots \\ N_{n+1/n}x_{m_\chi}^{(n+1)} \end{pmatrix} = U_{n+1} \begin{pmatrix} N_{n+1/n}y_1^{(n+1)} \\ \vdots \\ N_{n+1/n}y_{m_\chi}^{(n+1)} \end{pmatrix} = U_n \begin{pmatrix} N_{n+1/n}y_1^{(n+1)} \\ \vdots \\ N_{n+1/n}y_{m_\chi}^{(n+1)} \end{pmatrix} = \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_{m_\chi}^{(n)} \end{pmatrix}.
$$

Let μ_γ denote multiplication by γ in T_n^χ. By the definition of $A_\gamma^{(n),\chi}$, one has

$$
(17) \qquad\qquad \mu_\gamma\left(x_i^{(n)}\right) = \sum_{j=1}^{m_\chi} A_{\gamma,ij}^{(n),\chi} x_j^{(n)}, \quad \text{for all } n \geq 0 \text{ and } 1 \leq i \leq m_\chi.
$$

To prove commutativity of the left hand square of (15) one has to show

$$
(18) \qquad\qquad\qquad \varphi_{n+1/n}\left(A_{\gamma,ij}^{(n+1),\chi}\right) = A_{\gamma,ij}^{(n),\chi}.
$$

Since μ_γ is an R_n^χ–linear map, it follows from (16) and (17) that

$$
\begin{aligned}
\mu_\gamma\left(x_i^{(n)}\right) &= \mu_\gamma\left(N_{n+1/n}x_i^{(n+1)}\right) \\
&= N_{n+1/n}\left(\sum_{j=1}^{m_\chi} A_{\gamma,ij}^{(n+1),\chi}x_j^{(n+1)}\right) \\
&= \sum_{j=1}^{m_\chi} \varphi_{n+1/n}\left(A_{\gamma,ij}^{(n+1),\chi}\right) N_{n+1/n}\left(x_j^{(n+1)}\right) \\
&= \sum_{j=1}^{m_\chi} \varphi_{n+1/n}\left(A_{\gamma,ij}^{(n+1),\chi}\right) x_j^{(n)},
\end{aligned}
$$

and this, in turn, immediately implies (18).

Now, we start with an R_0^χ–basis $x_1^{(0)},\ldots,x_{m_\chi}^{(0)}$ for T_0^χ and use the procedure above inductively to construct R_n^χ–bases $x_1^{(n)},\ldots,x_{m_\chi}^{(n)}$ for T_n^χ so that (15) commutes, for all $n \geq 0$. Therefore, we can take a projective limit as $n \to \infty$ in (15). The Mittag-Leffler property (see [6, Prop. 9.1]) implies that we obtain an exact sequence of Λ^χ–modules

$$
(19) \qquad 0 \longrightarrow (\Lambda^\chi)^{m_\chi} \xrightarrow{\Phi_\gamma^{(\infty),\chi}} (\Lambda^\chi)^{m_\chi} \xrightarrow{\pi_\infty^\chi} T_\infty^\chi \longrightarrow 0,
$$

where $\Phi_\gamma^{(\infty),\chi} := \varprojlim_n \Phi_\gamma^{(n),\chi}$ and $\pi_\infty^\chi := \varprojlim_n \pi_n^\chi$. By (18), we may define the following matrix

$$
A_\gamma^{(\infty),\chi} := \{A_\gamma^{(n),\chi}\}_{n\geq 0} \in \varprojlim_n \mathrm{GL}_{m_\chi}(R_n^\chi) = \mathrm{GL}_{m_\chi}(R_\infty^\chi).
$$

Consequently, the map $\Phi_\gamma^{(\infty),\chi}$ has matrix $(1 - \gamma^{-1}A_\gamma^{(\infty),\chi})$ in the standard basis of $(\Lambda^\chi)^{m_\chi}$, for all characters χ as above.

If we take the direct sum of (19) over all χ we obtain the exact sequence of Λ–modules

$$
(20) \qquad 0 \longrightarrow \bigoplus_\chi (\Lambda^\chi)^{m_\chi} \xrightarrow{\Phi_\gamma^{(\infty)}} \bigoplus_\chi (\Lambda^\chi)^{m_\chi} \xrightarrow{\pi_\infty} T_\infty \longrightarrow 0,
$$

where $\Phi_\gamma^{(\infty)} := (\Phi_\gamma^{(\infty),\chi})_\chi$ and $\pi_\infty := (\pi_\infty^\chi)_\chi$. The exact sequence above shows that the Λ–module $T_\infty = T_p(M_S^{(\infty)})$ is finitely generated, of projective dimension at most 1.

Moreover, for all χ we have the following equalities

$$
(21) \qquad
\begin{aligned}
\mathrm{Fitt}_{\Lambda^\chi}\left(T_\infty^\chi\right) &\overset{(*)}{=} \det_{\Lambda^\chi}\left(1 - \gamma^{-1}A_\gamma^{(\infty),\chi}\right) \cdot \Lambda^\chi \\
&\overset{(**)}{=} \varprojlim_n \left(\det_{\Lambda_n^\chi}\left(1 - \gamma^{-1}A_\gamma^{(n),\chi}\right) \cdot \Lambda_n^\chi\right) \\
&\overset{(***)}{=} \varprojlim_n \mathrm{Fitt}_{\Lambda_n^\chi}\left(T_n^\chi\right).
\end{aligned}
$$

Above, equality (*) follows from (19), equality (**) follows from Lemma 3.10 and Corollary 3.12(1) below, and (***) follows from (14).

Now, we take the direct sum over all χ of equality (21) to obtain

$$\begin{aligned}
\text{Fitt}_\Lambda \left(T_\infty\right) &= \varprojlim_n \text{Fitt}_{\Lambda_n}\left(T_n\right) \\
&\stackrel{(*)}{=} \varprojlim_n \left(\Theta_{S,\Sigma}^{(n)}(\gamma^{-1}) \cdot \Lambda_n\right) \\
&\stackrel{(**)}{=} \Theta_{S,\Sigma}^{(\infty)}(\gamma^{-1}) \cdot \Lambda,
\end{aligned}$$

where (*) is one of the main results of Greither and Popescu (see Theorem 3.1(2) above) and (**) is Corollary 3.12(4) below.

Now, the fact that T_∞ is torsion and has projective dimension exactly equal to 1 over Λ follows from equality (21) above, Corollary 3.12(5), and Lemma 3.6.

This concludes the proof of Theorem 3.5, save for the technical results which imply the injectivity of the maps $\Phi_\gamma^{(n),\chi}$ and therefore the exactness of (14), as well as both equalities (**) above. We state and prove these technical results below.

Lemma 3.10. *Let $(R_m, \pi_{m,n})$ be a projective system of commutative rings and set $R_\infty := \varprojlim_m R_m$. Let $\alpha_\infty := \{\alpha_m\}_m \in R_\infty$ be a coherent sequence of non-zero divisors. Then*

$$\varprojlim_m \left(\alpha_m R_m\right) = \alpha_\infty R_\infty.$$

Proof. Let $\{\alpha_m r_m\}_m \in \varprojlim_m \alpha_m R_m$. Then, for $m > n$,

$$\alpha_n r_n = \pi_{m,n}(\alpha_m r_m) = \alpha_n \pi_{m,n}(r_m).$$

Since α_n is a non-zero divisor by assumption, this implies $r_n = \pi_{m,n}(r_m)$, and hence $\{\alpha_m r_m\}_m = \alpha_\infty r_\infty \in \alpha_\infty R_\infty$ with $r_\infty := \{r_m\}_m$. The opposite containment is obvious (and true without the assumption that the a_m are non-zero divisors). $\square$

Lemma 3.11. *Let R be R_n^χ or R_n. Let $f \in R[\gamma] \subseteq R[[\Gamma]]$ be a polynomial in γ such that the leading coefficient is a unit in R. Then f is a non-zero divisor in $R[[\Gamma]]$.*

Proof. We write

$$f = \lambda_d \gamma^d + \lambda_{d-1} \gamma^{d-1} + \ldots + \lambda_1 \gamma + \lambda_0$$

with $\lambda_j \in R$ and $\lambda_d \in R^\times$. We argue by contradiction and suppose that f is a zero divisor in $R[[\Gamma]]$. Then there exists $g \in R[[\Gamma]]$ such that $fg = 0$ and $g \neq 0$. Let $f = \{f_m\}_{m \in \mathbb{N}}$ and $g = \{g_m\}_{m \in \mathbb{N}}$ with $f_m, g_m \in R[\Gamma/\Gamma^m]$. Then there exists $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$

$$f_{Nm} g_{Nm} = 0, \quad f_{Nm} \neq 0 \neq g_{Nm}$$

in $R[\Gamma/\Gamma^{Nm}]$. In particular, for all $k \geq 0$, we have

(22) $$f_{Np^k} g_{Np^k} = 0, \quad f_{Np^k} \neq 0 \neq g_{Np^k}.$$

Write $N = Mp^a$ with $p \nmid M$. Since $\Gamma \simeq \widehat{\mathbb{Z}}$ (the profinite completion of $\mathbb{Z}$), we have the following group isomorphisms

$$\Gamma/\Gamma^{Np^k} = \Gamma/\Gamma^{Mp^{a+k}} \simeq \Gamma/\Gamma^M \times \Gamma/\Gamma^{p^{a+k}} \simeq \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/p^{a+k}\mathbb{Z}.$$

Further, we have a surjective topological group morphism

$$\Gamma \twoheadrightarrow \Gamma/\Gamma^M \times \varprojlim_k \Gamma/\Gamma^{p^{a+k}} \simeq \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}_p, \quad \gamma \to (\gamma_M, \gamma_p).$$

It follows that $\varprojlim_k R[\Gamma/\Gamma^{Np^k}] \simeq R[\Gamma/\Gamma^M][[t]]$, where γ_p maps to $1 + t$. Thus, we obtain a $\mathbb{Z}_p$–algebra homomorphism φ defined by the following composition of maps

$$\varphi: R[[\Gamma]] \xrightarrow{\pi_{M,p}} \varprojlim_k R[\Gamma/\Gamma^{Np^k}] \simeq R[\Gamma/\Gamma^M][[t]] \hookrightarrow \bigoplus_\psi \mathbb{Z}_p[\psi][[t]]$$

where ψ runs through the $\overline{\mathbb{Q}_p}$-valued characters of $(G_n \times \Gamma/\Gamma^M)$ modulo the action of $\mathrm{G}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ (and such that $\psi|_\Delta = \chi$ if $R = R_n^\chi$). Clearly, φ maps f to $(\psi(\lambda_d \gamma_M^d)(1+t)^d + \ldots)_\psi$. As $\lambda_d \in R^\times$ by assumption, we have $\psi(\lambda_d \gamma_M^d) \in \mathbb{Z}_p[\psi]^\times$, so the ψ-component of $\varphi(f)$ is non-zero, for all ψ. Since $\mathbb{Z}_p[\psi][[t]]$ is an integral domain, for all ψ, this implies that $\pi_{M,p}(f)$ is not a zero–divisor in $\varprojlim_k R[\Gamma/\Gamma^{Np^k}]$. However, this contradicts equalities (22), which concludes the proof of the Lemma.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3.12. *The following hold for all $n \geq 0$ and all $\chi \in \widehat{\Delta}$.*

(1) *The element $\det_{R_n^\chi[[\Gamma]]}\left(1 - \gamma^{-1} A_\gamma^{(n),\chi}\right)$ is a non-zero divisor in $R_n^\chi[[\Gamma]]$.*

(2) *The map $\Phi_\gamma^{(n),\chi}: R_n^\chi[[\Gamma]]^{m_\chi} \longrightarrow R_n^\chi[[\Gamma]]^{m_\chi}$ is injective.*

(3) *The element $\Theta_{S,\Sigma}^{(n)}(\gamma^{-1})$ is a non-zero divisor in $\Lambda_n = R_n[[\Gamma]]$.*

(4) *We have an equality $\Theta_{S,\Sigma}^{(\infty)}(\gamma^{-1}) \cdot \Lambda = \varprojlim_n \left(\Theta_{S,\Sigma}^{(n)}(\gamma^{-1}) \cdot \Lambda_n\right).$*

(5) *The element $\Theta_{S,\Sigma}^{(\infty)}(\gamma^{-1})$ is a non–zero divisor in Λ.*

Proof. (1) Observe that $\gamma^{m_\chi} \det_{R_n^\chi[[\Gamma]]}\left(1 - \gamma^{-1} A_\gamma^{(n),\chi}\right)$ is a polynomial in $R_n^\chi[\gamma]$ of degree m_χ and leading coefficient 1. Hence part (1) follows immediately from Lemma 3.11 above.

(2) is a consequence of (1) and the following fact: let R be a commutative ring, $A \in M_n(R)$ and suppose that $\det_R(A)$ is not a zero divisor. Then $A: R^n \longrightarrow R^n$ (defined with respect to the standard basis) is injective. Indeed, let A^* be the adjoint matrix. Then

$$AA^* = A^*A = \det_{\mathrm{R}}(\mathrm{A}) \cdot \mathrm{id},$$

and therefore $A^* \circ A$ is injective and, as a consequence, A is injective

(3) We apply results of [5], in particular Propositions 4.8 and 4.10. We can express the image $\Theta_{S,\Sigma}^{(n),\chi}(u)$ of $\Theta_{S,\Sigma}^{(n)}(u)$ in $R_n^\chi[u]$ as a product of two polynomials $P^\chi(u), Q^\chi(u) \in R_n^\chi[u]$, such that $Q^\chi(\gamma^{-1}) \in R_n^\chi[[\Gamma]]^\times$. It thus suffices to show that $P^\chi(\gamma^{-1})$ is a non-zero divisor. By [5, Prop. 4.8 a)] we have

$$P^\chi(u) = \det_{R_n^\chi}\left(1 - \gamma u \mid T_p(M_{\bar{S},\bar{\Sigma}}^{(n)})^\chi\right),$$

so the element $\gamma^{m_\chi} P^\chi(\gamma^{-1})$ is a polynomial with leading coefficient 1. Lemma 3.11 implies that $\Theta_{S,\Sigma}^{(n),\chi}(\gamma^{-1})$ is a non-zero divisor in Λ_n^χ, for all χ. Therefore, $\Theta_{S,\Sigma}^{(n)}(\gamma^{-1}) = (\Theta_{S,\Sigma}^{(n),\chi}(\gamma^{-1}))_\chi$ is a non–zero divisor in $\Lambda_n = \oplus_\chi \Lambda_n^\chi$.

(4) Apply part (3) combined with Lemmas 3.3 and 3.10.

(5) This follows immediately from (3), as $\Theta_{S,\Sigma}^{(\infty)}(\gamma^{-1}) = \varprojlim_n \Theta_{S,\Sigma}^{(n)}(\gamma^{-1})$ in $\Lambda = \varprojlim_n \Lambda_n$. $\square$

3.3. Co-descent to L_∞/k. In what follows, for every $n \geq 0$, we denote by $D^0(L_n)$ and $D_S^0(L_n)$ the $\mathbb{Z}[G_n]$–modules of divisors of degree 0 in L_n and divisors of degree 0 supported at primes above S in L_n, respectively. By $D_S(L_n)$ we denote the S–supported divisors of L_n of arbitrary degree. Note that the degree is computed relative to $\mathbb{F}_q$. Also, $U_S^{(n)}$ denotes the $\mathbb{Z}[G_n]$–module of S–units in L_n (i.e. elements $f \in L_n^\times$ whose divisor $\mathrm{div}(f)$ is in $D_S^0(L_n)$). Finally, $X_S^{(n)}$ denotes the $\mathbb{Z}[G_n]$–module of divisors of L_n supported at primes above S and of *formal degree* 0, i.e., formal sums $\sum_{v \in S(L_n)} n_v \cdot v$, with $n_v \in \mathbb{Z}$ and $\sum_v n_v = 0$.

By slightly generalizing the results in [4], our Proposition 4.5 and Remark 4.6 in the Appendix applied for $L = L_n$, for every $n \geq 0$, provide us with canonical exact sequences of $\mathbb{Z}_p[G_n]$–modules

$$(23) \qquad 0 \longrightarrow U_S^{(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow T_p(M_S^{(n)}) \xrightarrow{1-\gamma} T_p(M_S^{(n)}) \longrightarrow \triangledown_S^{(n)} \longrightarrow 0.$$

Here $\triangledown_S^{(n)} := T_p(M_S^{(n)})_\Gamma$ sits in a short exact sequence of $\mathbb{Z}_p[G_n]$–modules

$$(24) \qquad 0 \longrightarrow \mathrm{Pic}_S^0(L_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \triangledown_S^{(n)} \longrightarrow \widetilde{X}_S^{(n)} \longrightarrow 0,$$

and $\widetilde{X}_S^{(n)}$ (defined precisely in the appendix) sits itself in a short exact sequence

$$(25) \qquad 0 \to \mathbb{Z}_p/d_S^{(n)}\mathbb{Z}_p \to \widetilde{X}_S^{(n)} \to X_S^{(n)} \otimes \mathbb{Z}_p \to 0,$$

where $d_S^{(n)}\mathbb{Z} := \deg(D_S(L_n))$ and G_n acts trivially on $\mathbb{Z}_p/d_S^{(n)}\mathbb{Z}_p$. In particular, note that if $S(L_n)$ (the set of places of L_n sitting above places in S) contains a prime of degree coprime to p, then we have $\widetilde{X}_S^{(n)} = X_S^{(n)} \otimes \mathbb{Z}_p$.

Exact sequence (23) combined with Theorem 3.1 above, gives the following.

Corollary 3.13. *For all finite non-empty sets Σ with $S \cap \Sigma = \emptyset$ and all $n \in \mathbb{N}$ we have*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G_n]} \left(\triangledown_S^{(n)} \right) = \Theta_{S,\Sigma}^{(n)}(1) \cdot \mathbb{Z}_p[G_n].$$

Proof. Exact sequence (23) gives an isomorphism of $\mathbb{Z}_p[G_n]$–modules

$$\triangledown_S^{(n)} \simeq T_p(M_S^{(n)}) \otimes_{\mathbb{Z}_p[G_n][[\Gamma]]} \mathbb{Z}_p[G_n],$$

where $\mathbb{Z}_p[G_n]$ is viewed as a $\mathbb{Z}_p[G_n][[\Gamma]]$–algebra via the unique $\mathbb{Z}_p[G_n]$–algebra morphism $\pi : \mathbb{Z}_p[G_n][[\Gamma]] \twoheadrightarrow \mathbb{Z}_p[G_n]$ which takes $\gamma \to 1$. Since Fitting ideals commute with extension of scalars, this gives an equality of $\mathbb{Z}_p[G]$–ideals

$$\mathrm{Fitt}_{\mathbb{Z}_p[G_n]} \left(\triangledown_S^{(n)} \right) = \pi \left(\mathrm{Fitt}_{\mathbb{Z}_p[G_n][[\Gamma]]}(T_p(M_S^{(n)})) \right).$$

Now, the Corollary follows from Theorem 3.1. $\square$

In what follows, we set

$$U_S^{(\infty)} := \varprojlim_n (U_S^{(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_p), \quad \triangledown_S^{(\infty)} := \varprojlim_n \triangledown_S^{(n)},$$

where both limits are taken with respect to norm maps.

Lemma 3.14. *The sequence (23) stays exact when we pass to the limit, i.e.*

$$(26) \qquad 0 \longrightarrow U_S^{(\infty)} \longrightarrow T_p(M_S^{(\infty)}) \overset{1-\gamma}{\longrightarrow} T_p(M_S^{(\infty)}) \longrightarrow \triangledown_S^{(\infty)} \longrightarrow 0$$

is an exact sequence of Λ-modules.

Proof. We set $W^{(n)} := (1 - \gamma)T_p(M_S^{(n)})$. Then the functor $\varprojlim_n$ is exact on

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U_S^{(n+1)} \otimes_{\mathbb{Z}} \mathbb{Z}_p & \longrightarrow & T_p(M_S^{(n+1)}) & \longrightarrow & W^{(n+1)} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle N_{n+1/n}} & & \downarrow{\scriptstyle N_{n+1/n}} & & \downarrow{\scriptstyle N_{n+1/n}} & & \\
0 & \longrightarrow & U_S^{(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_p & \longrightarrow & T_p(M_S^{(n)}) & \longrightarrow & W^{(n)} & \longrightarrow & 0
\end{array}
$$

because $U_S^{(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a finitely generated $\mathbb{Z}_p$-module and therefore the projective system $\{U_S^{(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_p\}_n$ satisfies the Mittag-Leffler condition.

Since $N_{n+1/n}\colon T_p(M_S^{(n+1)}) \longrightarrow T_p(M_S^{(n)})$ is surjective, the map $N_{n+1/n}\colon W^{(n+1)} \longrightarrow W^{(n)}$ is also surjective. Hence, as above, $\varprojlim_n$ is exact on

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & W(n+1) & \longrightarrow & T_p(M_S^{(n+1)}) & \longrightarrow & \triangledown_S^{(n+1)} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle N_{n+1/n}} & & \downarrow{\scriptstyle N_{n+1/n}} & & \downarrow{\scriptstyle N_{n+1/n}} & & \\
0 & \longrightarrow & W^{(n)} & \longrightarrow & T_p(M_S^{(n)}) & \longrightarrow & \triangledown_S^{(n)} & \longrightarrow & 0
\end{array}
$$

We can now glue the two short exact sequences at the ∞-level. $\square$

The following is an equivariant Iwasawa main conjecture–type result along the Drinfeld module (geometric) tower L_∞/k, for the $\mathbb{Z}_p[[G_\infty]]$–module $\triangledown_S^{(\infty)}$, see Theorem 1.3 of the introduction.

Theorem 3.15 (EMC-II)**.** *For any finite, non-empty set Σ of primes in k, disjoint from S, the following hold.*

(1) $\triangledown_S^{(\infty)}$ *is a finitely generated, torsion $\mathbb{Z}_p[[G_\infty]]$–module of projective dimension 1.*

(2) $\mathrm{Fitt}_{\mathbb{Z}_p[[G_\infty]]}\left(\triangledown_S^{(\infty)}\right) = \Theta_{S,\Sigma}^{(\infty)}(1) \cdot \mathbb{Z}_p[[G_\infty]].$

Proof. Part (2) is an immediate consequence of exact sequence (26) and Theorem 3.5 above. (Repeat the arguments in the proof of Corollary 3.13.)

Part (1) is Proposition 3.24 in §3.5 below, which itself is a consequence of the fact that $\Theta_{S,\Sigma}^{(\infty)}(1)$ is a non–zero divisor in $\mathbb{Z}_p[[G_\infty]]$, as proved in Proposition 3.22 below. $\square$

3.4. **Results on ideal class groups.** We conclude this section by deriving an Iwasawa main conjecture in the spirit of [1, Th. 1.4] for the classical $\mathbb{Z}_p[[G_\infty]]$–module

$$\mathfrak{X}_p^{(\infty)} := \varprojlim_n \left(\mathrm{Pic}^0(L_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p\right),$$

where the projective limit is taken with respect to the usual norm maps.

As in (12) we let Δ denote the maximal subgroup of G_0 whose order is not divisible by p. Since $G(L_\infty/L_0)$ is a pro–p–group, we have $G_\infty \simeq \Delta \times G_\infty^{(p)}$, where $G_\infty^{(p)}$ is the maximal pro–p subgroup of G_∞. As a consequence, we can view the element $e_\Delta := \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \delta$ as an idempotent in $\mathbb{Z}_p[[G_\infty]]$. This allows us to define the functor

$$M \mapsto M^\sharp := (1 - e_\Delta) \cdot M$$

from the category of $\mathbb{Z}_p[[G_\infty]]$–modules to the category of modules over the quotient ring

$$\mathbb{Z}_p[[G_\infty]]^\sharp = (1 - e_\Delta)\mathbb{Z}_p[[G_\infty]].$$

Note that since $M = e_\Delta M \oplus M^\sharp$, for every M as above, the functor $M \mapsto M^\sharp$ is exact.

The study of $\mathfrak{X}_p^{(\infty)}$ requires some additional hypotheses. We list them below.

(a) $\mathfrak{f} = \mathfrak{e}$.
(b) $\mathfrak{p}$ does not split in $H_\mathfrak{e}/k$.
(c) $p \nmid [H_\mathfrak{e} : k] = h_k d_\infty$.
(d) $p \nmid \deg(\mathfrak{p})$.

Note that (a), (b) and (c) are satisfied in the basic example of the Carlitz module, see §2.3.

Theorem 3.16 (EMC-III). *Under hypotheses (a)–(d), the following hold for $S = \{\mathfrak{p}\}$ and all nonempty sets Σ, disjoint from S.*

(1) *$\mathfrak{X}_p^{(\infty)}$ is a torsion $\mathbb{Z}_p[[G_\infty]]$–module of projective dimension 1.*

(2) *$\mathrm{Fitt}_{\mathbb{Z}_p[[G_\infty]]} \left(\mathfrak{X}_p^{(\infty)} \right) = \Theta_{S,\Sigma}^{(\infty)}(1) \cdot \mathbb{Z}_p[[G_\infty]].$*

Proof. Since $L_n = H_{\mathfrak{p}^{n+1}}$ is unramified outside $\mathfrak{p}$, the set $S = \{\mathfrak{p}\}$ satisfies all the desired requirements. In general, each divisor of $\mathfrak{p}$ in $H_\mathfrak{e}$ is totally ramified in $H_{\mathfrak{p}^{n+1}}/H_\mathfrak{e}$. Hence, hypotheses (a)–(b) imply that $D_S^0(L_n) = 0$ and $X_S^{(n)} = 0$. Further, hypotheses (c)–(d) imply that if $\mathfrak{p}_n$ is the unique prime in L_n sitting above $\mathfrak{p}$, then $d_S^{(n)} = \deg(\mathfrak{p}_n) = [H_\mathfrak{e} : k] \cdot \deg(\mathfrak{p})$ is not divisible by p. Consequently, (24) and (25) give us

$$\nabla_S^{(n)} = \mathrm{Pic}^0(L_n) \otimes \mathbb{Z}_p, \qquad \nabla_S^{(\infty)} = \mathfrak{X}_p^{(\infty)},$$

and the result follows from Theorem 3.15 above. $\qquad\square$

Under the milder hypotheses (a)–(b), both satisfied in the basic case of the Carlitz module, a similar result holds, away from the trivial character of Δ.

Theorem 3.17 (EMC $-$ III$^\sharp$). *Under hypotheses (a)–(b), the following hold for $S = \{\mathfrak{p}\}$ and all nonempty sets Σ, disjoint from S.*

(1) *$\mathfrak{X}_p^{(\infty),\sharp}$ is a torsion $\mathbb{Z}_p[[G_\infty]]^\sharp$–module of projective dimension 1.*

(2) *$\mathrm{Fitt}_{\mathbb{Z}_p[[G_\infty]]^\sharp} \left(\mathfrak{X}_p^{(\infty),\sharp} \right) = \Theta_{S,\Sigma}^{(\infty)}(1) \cdot \mathbb{Z}_p[[G_\infty]]^\sharp.$*

Proof. Once again, we have $X_S^{(n)} = 0$. Since $(\mathbb{Z}_p/d_S^{(n)}\mathbb{Z}_p)^\sharp = 0$ (as Δ acts trivially on $\mathbb{Z}_p/d_S^{(n)}\mathbb{Z}_p$), when applying the exact functor $^\sharp$ to exact sequences (24) and (25), we obtain

$$\nabla_S^{(n),\sharp} = (\mathrm{Pic}^0(L_n) \otimes \mathbb{Z}_p)^\sharp, \qquad \nabla_S^{(\infty),\sharp} = \mathfrak{X}_p^{(\infty),\sharp},$$

and the result follows by projecting the equality in Theorem 3.15 onto $\mathbb{Z}_p[[G_\infty]]^\sharp$. $\qquad\square$

Remark 3.18. In this remark we use the assumptions of [2] and [3]. More specifically, we assume $\mathfrak{f} = \mathfrak{e}$, $d_\infty = 1$ and $p \nmid [H_\mathfrak{e} : k]$. Note that, for example, these assumptions hold in the case of the Carlitz module which is studied in [1].

We let $G_\mathfrak{p}(L_0/k)$ and $I_\mathfrak{p}(L_0/k)$ denote the decomposition, respectively inertia group associated to $\mathfrak{p}$ in L_0/k. We observe that $\Delta = \mathrm{G}(L_0/k)$ has order prime to p and let $\chi \in \hat{\Delta}$ be a character, such that $\chi|_{G_\mathfrak{p}(L_0/k)}$ is non-trivial. Since $I_\mathfrak{p}(L_0/k) = \mathrm{G}(L_0/H_\mathfrak{e}) \subseteq G_\mathfrak{p}(L_0/k)$, the set of such characters χ includes the characters of type 2 as defined in [2, Def. 3.1] or [3, Def. 2.3.6]. Since we only consider real ray–class fields, we do not see characters χ of type 1, as defined in loc.cit. Then, for all $n \in \mathbb{N} \cup \{\infty\}$ and $S = \{\mathfrak{p}\}$, we have

$$\begin{aligned} \left(X_S^{(n)} \otimes_\mathbb{Z} \mathbb{Z}_p\right)^\chi &= (D_S(L_n) \otimes_\mathbb{Z} \mathbb{Z}_p)^\chi \\ &\simeq \mathbb{Z}_p[\mathrm{G}(L_n/k)/G_\mathfrak{p}(L_n/k)]^\chi \\ &\simeq \mathbb{Z}_p[\Delta/G_\mathfrak{p}(L_0/k)]^\chi = 0. \end{aligned}$$

As a consequence, we have $\bigtriangledown_S^{(\infty),\chi} = \mathfrak{X}_p^{(\infty),\chi}$ and our Theorem 3.15(2) implies that

$$\mathrm{Fitt}_{\mathbb{Z}_p(\chi)[[\Gamma_\infty]]}\left(\mathfrak{X}_p^{(\infty),\chi}\right) = \Theta_{S,\Sigma}^{(\infty)}(1,\chi) \cdot \mathbb{Z}_p(\chi)[[\Gamma_\infty]],$$

for all characters χ as above. Thus we recover the central results of [1, Thm. 1.1], [2] and [3, Thm. 2.4.8] restricted to the real Iwasawa towers considered in loc.cit.

3.5. $\Theta_{S,\Sigma}^{(\infty)}(1)$ **is a non-zero divisor.** The goal of this section is a proof of part (1) of Theorem 3.15 above. We will first establish a structure theorem for the Iwasawa algebra $\mathbb{Z}_p[[G_\infty]]$ whose proof will be based on the following result on pro-p groups.

Theorem 3.19 (Theorem 3.1 of [9]). *Let $\mathcal{G}$ be a pro-p group with countable (topological) basis, whose torsion subroup $t(\mathcal{G})$ has bounded exponent. Then $t(\mathcal{G})$ is a closed subgroup of $\mathcal{G}$ and we have an isomorphism of topological groups*

$$\mathcal{G} \simeq t(\mathcal{G}) \times \mathbb{Z}_p^X,$$

where X is a cardinal in the set $\mathbb{N} \cup \{\aleph_0\}$.

Here is the promised structure theorem for the Iwasawa algebra $\mathbb{Z}_p[[G_\infty]]$.

Proposition 3.20. *The following hold.*

(1) *There are closed subgroups $\widetilde{G_0}$ and $\widetilde{\Gamma_\infty}$ of G_∞, such that*

$$G_\infty = \widetilde{G_0} \times \widetilde{\Gamma_\infty},$$

with $\widetilde{\Gamma_\infty} \simeq \mathbb{Z}_p^{\aleph_0}$ (topologically), $[\Gamma_\infty : \Gamma_\infty \cap \widetilde{\Gamma_\infty}] < \infty$, $[\widetilde{\Gamma_\infty} : \Gamma_\infty \cap \widetilde{\Gamma_\infty}] < \infty$, and $\widetilde{G_0}$ is isomorphic to a subgroup of G_0.

(2) *There is an injective morphism of topological $\mathbb{Z}_p$–algebras*

$$\mathbb{Z}_p[[G_\infty]] \simeq \mathbb{Z}_p[\widetilde{G_0}][[\widetilde{\Gamma_\infty}]] \hookrightarrow \bigoplus_{\rho \in \widehat{\widetilde{G_0}}(\mathbb{Q}_p)} \mathbb{Z}_p(\rho)[[\widetilde{\Gamma_\infty}]] \simeq \bigoplus_{\rho \in \widehat{\widetilde{G_0}}(\mathbb{Q}_p)} \mathbb{Z}_p(\rho)[[X_1, X_2, \dots]],$$

where the injective map in the middle is given by the usual ρ–evaluation maps.

Proof. Part (2) is a clear consequence of part (1) and Proposition 2.9. For the proof of (1), note that, with notations as in §3.2 above, if we let $P_\infty := \varprojlim_n P_n$, then we have

$$(27) \qquad G_\infty \simeq P_\infty \times \Delta, \qquad P_\infty / \Gamma_\infty \simeq P_0.$$

Recall that Δ is the complement of the p–Sylow subgroup P_n of G_n, for all $n \geq 0$. Since Γ_∞ is torsion free, the second isomorphism above implies that $t(P_\infty)$ is isomorphic to a subgroup of P_0, therefore it is finite and, obviously, of bounded exponent. Since Γ_∞ has countable basis, the second isomorphism above implies that P_∞ has countable basis as well. Consequently, Theorem 3.19 applied to $\mathcal{G} := P_\infty$ gives a topological isomorphism

$$(28) \qquad P_\infty \simeq t(P_\infty) \times \widetilde{\Gamma_\infty},$$

where $\widetilde{\Gamma_\infty} \simeq \mathbb{Z}_p^{\aleph_0}$. Consequently, we have

$$G_\infty = \widetilde{G_0} \times \widetilde{\Gamma_\infty}, \qquad \text{for } \widetilde{G_0} := t(G_\infty) \simeq t(P_\infty) \times \Delta.$$

Obviously, $\widetilde{G_0}$ is isomorphic to a subgroup of $G_0 \simeq P_0 \times \Delta$. The fact that $[\Gamma_\infty : \Gamma_\infty \cap \widetilde{\Gamma_\infty}] < \infty$ and $[\widetilde{\Gamma_\infty} : \Gamma_\infty \cap \widetilde{\Gamma_\infty}] < \infty$ follows immediately from (27) and (28). $\qquad\square$

Remark 3.21. From now on, we let

$$\varphi = (\varphi_\rho)_\rho : \mathbb{Z}_p[[G_\infty]] \simeq \mathbb{Z}_p[\widetilde{G_0}][[\widetilde{\Gamma_\infty}]] \hookrightarrow \bigoplus_{\rho \in \widehat{\widetilde{G_0}}(\mathbb{Q}_p)} \mathbb{Z}_p(\rho)[[\widetilde{\Gamma_\infty}]] =: \overline{\mathbb{Z}_p[[G_\infty]]}$$

denote the character evaluation map described above. Proposition 2.9(2)–(3) implies that the direct summands of $\overline{\mathbb{Z}_p[[G_\infty]]}$ are integral, normal domains. (Notice that $\overline{\mathbb{Z}_p[[G_\infty]]}$ is the integral closure of $\mathbb{Z}_p[[G_\infty]]$ in its total ring of fractions, which justifies the notation.)

Proposition 3.22. $\Theta_{S,\Sigma}^{(\infty)}(1)$ *is a non-zero divisor in* $\overline{\mathbb{Z}_p[[G_\infty]]}$ *and therefore in* $\mathbb{Z}_p[[G_\infty]]$.

Proof. Proposition 3.20(2) implies that the statement to be proved is equivalent to

$$\varphi_\rho(\Theta_{S,\Sigma}^{(\infty)}(1)) \neq 0 \text{ in } \mathbb{Z}_p(\rho)[[\widetilde{\Gamma_\infty}]],$$

for all characters $\rho \in \widehat{\widetilde{G_0}}$. Of course, this is equivalent to proving that for every ρ as above, there exists a character ψ of $\widetilde{\Gamma_\infty}$, of open kernel (so, of finite order), such that

$$\psi(\varphi_\rho(\Theta_{S,\Sigma}^{(\infty)}(1)) \neq 0 \text{ in } \mathbb{Z}_p(\rho\psi).$$

However, from the definition of $\Theta_{S,\Sigma}^{(\infty)}$, for all ρ and ψ as above, we have an equality

$$\psi(\varphi_\rho(\Theta_{S,\Sigma}^{(\infty)}(1)) = L_{S,\Sigma}((\rho\psi)^{-1}, 0),$$

where $\rho\psi$ is viewed as a complex–valued character of the finite quotient

$$\widetilde{G_0} \times (\widetilde{\Gamma_\infty}/\ker\psi) = G_\infty/\ker\psi$$

of G_∞ (under a fixed field isomorphism $\mathbb{C}_p \simeq \mathbb{C}$) and $L_{S,\Sigma}(\rho\psi, s)$ is the complex–valued Artin L–function (S–imprimitive and Σ–completed) associated to $\rho\psi$. (See equalities (7) above.)

Now, the following Lemma is a well–known description of the order of vanishing at $s = 0$ of the Artin L–functions in question. For the number field case of this result, see [17, Ch. I, Prop. 3.4] and for the function field case, relevant in our context, see [14, Sec. 2.2].

Lemma 3.23. *If χ is a non–trivial character of G_∞ with open kernel, then*

$$\mathrm{ord}_{s=0} L_{S,\Sigma}(\chi, s) = \mathrm{card}\, \{v \in S \mid \chi(G_v(L_\infty/k)) = 1\},$$

where $G_v(L_\infty/k)$ denotes the decomposition group of v inside $G_\infty = G(L_\infty/k)$.

Consequently, we claim that it suffices to find a finite subextension L/K of $L_\infty/L_\infty^{\widetilde{\Gamma_\infty}}$ and a character χ of $G(L/K)$, such that the following conditions are simultaneously satisfied.

(A1) $\widetilde{L_0} := L_\infty^{\widetilde{\Gamma_\infty}} \subseteq K \subseteq L \subseteq L_\infty$ and $L/\widetilde{L_0}$ finite.
(A2) $\chi(G_v(L/K)) \neq \{1\}$, for all $v \in S$.

Indeed, if we construct an L/K and χ as above, then for any given character ρ of $\widetilde{G}_0 \simeq G(\widetilde{L_0}/k)$, we take any character ψ of $G(L/\widetilde{L_0})$, such that $\psi \mid_{G(L/K)} = \chi$. Now, $\rho\psi$ is a character of $G(L/k)$ which satisfies the property

$$\rho\psi(G_v(L/K)) = \psi(G_v(L/K)) = \chi(G_v(L/K)) \neq \{1\}, \qquad \text{for all } v \in S.$$

Since $G_v(L/K) \subseteq G_v(L/k)$, for all $v \in S$, Lemma 3.23 gives us the desired nonvanishing

$$\psi(\varphi_\rho(\Theta_{S,\Sigma}^{(\infty)}(1))) = L_{S,\Sigma}((\rho\psi)^{-1}, 0) \neq 0.$$

Now, since $[\widetilde{\Gamma_\infty} : \widetilde{\Gamma_\infty} \cap \Gamma_\infty] < \infty$, the existence of L/K satisfying conditions (A1)-(A2) above is ensured if we can find two integers $m > n$ and a character χ of $G(L_m/L_n)$, such that

(B1) n is large enough, so that $L_\infty^{\Gamma_\infty \cap \widetilde{\Gamma_\infty}} \subseteq L_n$. (Note that $\widetilde{L_0} \subseteq L_\infty^{\Gamma_\infty \cap \widetilde{\Gamma_\infty}}$.)
(B2) $\chi(G_v(L_m/L_n)) \neq \{1\}$, for all primes $v \in S$.

Now, we proceed to constructing m and n as above. First, we fix an $n \geq 0$, large enough so that (B1) is satisfied. Now, we apply Proposition 2.11 to get topological group isomorphisms

$$(29) \qquad G_{\mathfrak{f}}(L_\infty/L_n) = \prod_{v|\mathfrak{f}} G_v(L_\infty/L_n) \simeq \prod_{v|\mathfrak{f}} \mathbb{Z}_p,$$

where the notations are as in loc.cit. Since the p–adic and the profinite topologies on $G_{\mathfrak{f}}(L_\infty/L_n)$ coincide, there exists an $m > n$, such that

$$(30) \qquad G_{\mathfrak{f}}(L_\infty/L_n) \cap G(L_\infty/L_m) \subseteq p \cdot G_{\mathfrak{f}}(L_\infty/L_n).$$

We let $G_{\mathfrak{f}}(L_m/L_n)$ denote the subgroup of $G(L_m/L_n)$ generated by the decomposition groups $G_v(L_m/L_n)$, for all $v|\mathfrak{f}$. From the definitions, we have a group morphism

$$G_{\mathfrak{f}}(L_m/L_n) \simeq \frac{G_{\mathfrak{f}}(L_\infty/L_n)}{G_{\mathfrak{f}}(L_\infty/L_n) \cap G(L_\infty/L_m)} \twoheadrightarrow \frac{G_{\mathfrak{f}}(L_\infty/L_n)}{p \cdot G_{\mathfrak{f}}(L_\infty/L_n)} = \prod_{v|\mathfrak{f}} \frac{G_v(L_\infty/L_n)}{p \cdot G_v(L_\infty/L_n)},$$

where the isomorphism to the left is induced by Galois restriction, the surjection is induced by the inclusion (30) and the equality is a consequence of (29).

It is easily seen that for all $v|\mathfrak{f}$ the above morphism maps $G_v(L_m/L_n)$ onto the quotient $G_v(L_\infty/L_n)/pG_v(L_\infty/L_n)$ which by Proposition 2.11 is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Consequently, there is a character ψ of $G_{\mathfrak{f}}(L_m/L_n)$, such that $\psi(G_v(L_m/L_n)) = \{\zeta \in \mathbb{C}_p \mid \zeta^p = 1\}$ for all $v|\mathfrak{f}$. Now, take any character χ of $G(L_m/L_n)$ which equals ψ when restricted to $G_{\mathfrak{f}}(L_m/L_n)$. This character obviously satisfies (B2) for all $v|\mathfrak{f}$. Since it is non–trivial on $G(L_m/L_n)$ and $G_{\mathfrak{p}}(L_m/L_n) = G(L_m/L_n)$ (recall that L_m/L_n is totally ramified at the $\mathfrak{p}$–adic primes), the character χ also satisfies (B2) for $v = \mathfrak{p}$. This concludes the proof of Proposition 3.22. $\qquad \square$

We conclude this section with a corollary to Proposition 3.22.

Proposition 3.24. *The $\mathbb{Z}_p[[G_\infty]]$–module $\triangledown_S^{(\infty)}$ is finitely generated, torsion, and of projective dimension 1.*

Proof. We will use the notations in the proof of Theorem 3.5. In particular, note that

$$\Lambda^\chi/(1-\gamma) \simeq \mathbb{Z}_p(\chi)[[P_\infty]],$$

for all $\overline{\mathbb{Q}_p}$–valued characters χ of Δ. Consequently, the exact sequence (20) leads to the following commutative diagram of Λ–modules.

$$
\begin{array}{ccccccccc}
& & & & & & & & U_S^{(\infty)} \\
& & & & & & & & \downarrow \\
0 & \longrightarrow & \bigoplus_\chi (\Lambda^\chi)^{m_\chi} & \xrightarrow{\Phi_\gamma^{(\infty)}} & \bigoplus_\chi (\Lambda^\chi)^{m_\chi} & \xrightarrow{\pi_\infty} & T_p(M_S^{(\infty)}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle \gamma-1} & & \\
0 & \longrightarrow & \bigoplus_\chi (\Lambda^\chi)^{m_\chi} & \xrightarrow{\Phi_\gamma^{(\infty)}} & \bigoplus_\chi (\Lambda^\chi)^{m_\chi} & \xrightarrow{\pi_\infty} & T_p(M_S^{(\infty)}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \bigoplus_\chi \mathbb{Z}_p(\chi)[[P_\infty]]^{m_\chi} & & \bigoplus_\chi \mathbb{Z}_p(\chi)[[P_\infty]]^{m_\chi} & & \triangledown_S^{(\infty)} & &
\end{array}
$$

where the right vertical exact sequence is given by Lemma 3.14. The snake lemma applied to the diagram above gives the exact sequence of $\mathbb{Z}_p[[G_\infty]]$–modules

$$\bigoplus_\chi \mathbb{Z}_p(\chi)[[P_\infty]]^{m_\chi} \xrightarrow{\overline{\Phi_\gamma^{(\infty)}}} \bigoplus_\chi \mathbb{Z}_p(\chi)[[P_\infty]]^{m_\chi} \longrightarrow \triangledown_S^{(\infty)} \longrightarrow 0,$$

where $\overline{\Phi_\gamma^{(\infty)}} := \Phi_\gamma^{(\infty)} \mod (\gamma-1)$. It follows that

$$\mathrm{Fitt}_{\mathbb{Z}_p[[G_\infty]]}\left(\triangledown_S^{(\infty)}\right) = \det\nolimits_{\mathbb{Z}_p[[G_\infty]]}\left(\overline{\Phi_\gamma^{(\infty)}}\right) \cdot \mathbb{Z}_p[[G_\infty]].$$

Combined with Theorem 3.15(2), the equality above shows that the element $\det_{\mathbb{Z}_p[[G_\infty]]}\left(\overline{\Phi_\gamma^{(\infty)}}\right)$ differs from $\Theta_{S,\Sigma}^{(\infty)}(1)$ by a unit in $\mathbb{Z}_p[[G_\infty]]$. By Propositon 3.22, the element $\Theta_{S,\Sigma}^{(\infty)}(1)$ is a non-zero divisor in $\mathbb{Z}_p[[G_\infty]]$. Therefore, $\det_{\mathbb{Z}_p[[G_\infty]]}\left(\overline{\Phi_\gamma^{(\infty)}}\right)$ is a non-zero divisor in $\mathbb{Z}_p[[G_\infty]]$ as well. By a standard argument (using the adjoint matrix χ–componentwise, see proof of Cor. 3.12(2)) we see that $\overline{\Phi_\gamma^{(\infty)}}$ is injective, hence the sequence of $\mathbb{Z}_p[[G_\infty]]$–modules

$$0 \longrightarrow \bigoplus_\chi \mathbb{Z}_p(\chi)[[P_\infty]]^{m_\chi} \xrightarrow{\overline{\Phi_\gamma^{(\infty)}}} \bigoplus_\chi \mathbb{Z}_p(\chi)[[P_\infty]]^{m_\chi} \longrightarrow \triangledown_S^{(\infty)} \longrightarrow 0$$

is exact. Consequently, the $\mathbb{Z}_p[[G_\infty]]$–module $\triangledown_S^{(\infty)}$ is finitely generated, of projective dimension at most one. Further, the fact that $\triangledown_S^{(\infty)}$ is torsion and of projective dimension exactly 1 as a $\mathbb{Z}_p[[G_\infty]]$–module follows immediately from Lemma 3.6. $\square$

4. APPENDIX (p–ADIC RITTER–WEISS MODULES AND TATE SEQUENCES FOR SMALL S)

Let L be a finite, separable extension of $\mathbb{F}_q(t)$. Denote by Z a smooth, projective curve defined over $\mathbb{F}_q$, whose field of rational functions is isomorphic to L. We let $\overline{Z} := Z \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, $\Gamma := G(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ and let γ be the q–power arithmetic Frobenius automorphism, viewed as a canonical topological generator of Γ. Note that $\overline{Z}$ may not be connected. Consequently, $\overline{L} := L \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ (the $\overline{\mathbb{F}_q}$–algebra of rational functions on $\overline{Z}$) could be a finite direct sum of isomorphic fields (the fields of rational functions of the connected components of $\overline{Z}$.)

Next, we consider a finite, non–empty set S of closed points on Z and let $\overline{S}$ be the set of closed points on $\overline{Z}$ sitting above points in S. We let $\mathrm{Div}^0_{\overline{S}}(\overline{L})$ (respectively $\mathrm{Div}_{\overline{S}}(\overline{L})$) and $\mathrm{Div}^0_S(L)$ (respectively $\mathrm{Div}_S(L)$) denote the divisors of degree 0 (respectively arbitrary degree) on $\overline{Z}$ and Z, supported at $\overline{S}$ and S, respectively. Note that the degree of a divisor on Z, denoted by deg, is computed relative to the field of definition $\mathbb{F}_q$. Also, the degree of a divisor on $\overline{Z}$ is in fact a multidegree, computed on each connected component on $\overline{Z}$ separately. Further, $X_S(L)$ denotes the S–supported divisors on Z of arbitrary formal degree, denoted below fdeg. Also, $U_S(L)$ denotes the group of S–units inside $L^\times$ and

$$\mathrm{Pic}^0_S(L) := \frac{\mathrm{Pic}^0(L)}{\widehat{\mathrm{Div}^0_S(L)}} = \frac{\mathrm{Div}^0(L)}{\mathrm{Div}^0_S(L) + \mathrm{div}(L^\times)},$$

is the S–Picard group associated to L, obtained by taking the quotient of the usual Picard group $\mathrm{Pic}^0(L)$ by the subgroup $\widehat{\mathrm{Div}^0_S(L)}$ of classes of all S–supported divisors of degree 0.

Finally, we let M_S denote the Picard 1–motive associated as in [5] to the data $(\overline{Z}, \overline{\mathbb{F}_q}, \overline{S}, \emptyset)$. As usual, $T_p(M_S)^\Gamma$ and $T_p(M_S)_\Gamma$ denote the Γ–invariants, respectively Γ–coinvariants of the p–adic Tate module of M_S. In what follows, if N is a $\mathbb{Z}$–module, we let $N_p := N \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Definition 4.1. The set S is called p–large if the following are satisfied.
  (1) $\mathrm{Pic}^0_S(L)_p = 0$.
  (2) S contains at least one place of degree (relative to $\mathbb{F}_q$) coprime to p.

Remark 4.2. It is easily seen that S is p–large if and only if $\mathrm{Pic}_S(L)_p = 0$, where

$$\mathrm{Pic}_S(L) := \frac{\mathrm{Div}(L)}{\mathrm{Div}_S(L) + \mathrm{div}(L^\times)}$$

is the quotient of the full Picard group $\mathrm{Pic}(L)$ of L by its subgroup of S–supported divisor classes. This is perhaps a more natural definition, but we prefer to use the definition above because $\mathrm{Pic}^0(L)$ (as opposed to $\mathrm{Pic}(L)$) is much more naturally related to the 1–motive M_S.

The following result was obtained in [4]. (See Proposition 1.1 in loc.cit.)

Proposition 4.3 (Greither–Popescu, [4]). *If S is p–large, then the following hold.*
  (1) *There is a canonical isomorphism $T_p(M_S)^\Gamma \simeq U_S(L)_p$.*
  (2) *There is a canonical isomorphism $T_p(M_S)_\Gamma \simeq X_S(L)_p$.*

Remark 4.4. In fact, in [4], the authors describe the modules $T_p(M_{S,T})^\Gamma$ and $T_p(M_{S,T})_\Gamma$, where $M_{S,T}$ is the Picard 1–motive associated to $(\overline{Z}, \overline{\mathbb{F}_q}, \overline{S}, \overline{T})$, where T is a finite, non–empty set of

closed points on Z, disjoint from S. However, by [5, Rem. 2.7] and the proof of [4, Lemma 3.2], we have for any such T equalities

$$T_p(M_{S,T}) = T_p(M_S), \qquad U_{S,T}(L)_p = U_S(L)_p,$$

where $U_{S,T}(L)$ is the group of S–units in $L^\times$, congruent to 1 modulo all primes in T.

The goal of this Appendix is to remove the hypothesis "S is p–large" in the Proposition above. More precisely, we sketch the proof of the following.

Proposition 4.5. *With notations as above, the following hold for all sets S.*
  (1) *There is a canonical isomorphism $T_p(M_S)^\Gamma \simeq U_S(L)_p$.*
  (2) *There are canonical exact sequences of $\mathbb{Z}_p$–modules:*

$$0 \to \mathrm{Pic}_S^0(L)_p \to T_p(M_S)_\Gamma \to \widetilde{X}_S(L) \to 0,$$
$$0 \to \mathbb{Z}_p/d_S\mathbb{Z}_p \to \widetilde{X}_S(L) \to X_S(L)_p \to 0,$$

  where $d_S\mathbb{Z} = \deg(\mathrm{Div}_S(L))$ and $\widetilde{X}_S(L) := (\mathrm{Div}_{\overline{S}}^0(\overline{L})_p)_\Gamma$. In particular, if S contains a prime of degree not divisible by p, then $\widetilde{X}_S(L) = X_S(L)_p$.

Proof. (Sketch) We will give only a brief sketch of the proof, as the techniques and main ideas are borrowed from [4]. First, we consider the exact sequence of $\mathbb{Z}_p[[\Gamma]]$–modules

$$0 \to \mathrm{Div}_{\overline{S}}^0(\overline{L})_p \to \mathrm{Div}_{\overline{S}}(\overline{L})_p \xrightarrow{\deg} \mathbb{Z}_p \to 0$$

and take Γ–invariants and Γ–coinvariants to obtain a long exact sequence

$$0 \to \mathrm{Div}_S^0(L)_p \to \mathrm{Div}_S(L)_p \xrightarrow{\deg} \mathbb{Z}_p \to (\mathrm{Div}_{\overline{S}}^0(\overline{L})_p)_\Gamma \to \mathrm{Div}_S(L)_p \xrightarrow{\mathrm{fdeg}} \mathbb{Z}_p \to 0.$$

The fact that the Γ–invariant of the complex $[\mathrm{Div}_{\overline{S}}(\overline{L})_p \xrightarrow{\deg} \mathbb{Z}_p]$ is $[\mathrm{Div}_S(L)_p \xrightarrow{\deg} \mathbb{Z}_p]$ and its Γ–coinvariant is $[\mathrm{Div}_S(L)_p \xrightarrow{\mathrm{fdeg}} \mathbb{Z}_p]$ follows immediately from the definitions and is explained in §2 of [4]. Now, in the long exact sequence above, we have

$$\ker(\mathrm{fdeg}) = X_S(L)_p, \qquad \mathrm{coker}(\deg) = \mathbb{Z}_p/d_S\mathbb{Z}_p.$$

Therefore, if we let $\widetilde{X}_S(L) := (\mathrm{Div}_{\overline{S}}^0(\overline{L})_p)_\Gamma$, we have a canonical exact sequence

(31) $$0 \to \mathbb{Z}_p/d_S\mathbb{Z}_p \to \widetilde{X}_S(L) \to X_S(L)_p \to 0.$$

If J denotes the Jacobian of $\overline{Z}$, there is a canonical exact sequence of $\mathbb{Z}_p$–modules

$$0 \to T_p(J) \to T_p(M_S) \to \mathrm{Div}_{\overline{S}}^0(\overline{L})_p \to 0.$$

(See §2 of [5] for the exact sequence above.) Since we have a canonical isomorphism

$$T_p(J)_\Gamma \simeq \mathrm{Pic}^0(L)_p$$

(see Corollary 5.7 in [5]) and $T_p(J)$ is $\mathbb{Z}_p$–free of finite rank, we also have

$$T_p(J)^\Gamma = 0.$$

Therefore, when taking Γ–invariants and Γ–coinvariants in the above exact sequence, we obtain a canonical long exact sequence of $\mathbb{Z}_p$–modules

$$0 \to T_p(M_S)^\Gamma \to \mathrm{Div}_S^0(L)_p \xrightarrow{\delta} \mathrm{Pic}^0(L)_p \to T_p(M_S)_\Gamma \to \widetilde{X}_S(L) \to 0,$$

where the connecting morphism δ is the usual divisor–class map. (See [4, §1] for this fact.) Since there is a canonical isomorphism $U_S(L)_p \simeq \ker(\delta)$, where $U_S(L)_p$ injects into $\mathrm{Div}_S^0(L)_p$ via the divisor map, we obtain a canonical isomorphism of $\mathbb{Z}_p$–modules

$$T_p(M_S)^\Gamma \simeq U_S(L)_p,$$

which concludes the proof of part (1) of the Proposition.

To conclude the proof of part (2), observe that, by definition, we have $\mathrm{coker}(\delta) = \mathrm{Pic}_S^0(L)_p$. Therefore, the last four non–zero terms of the long exact sequence above lead to a canonical short exact sequence of $\mathbb{Z}_p$–modules

$$(32) \qquad\qquad 0 \to \mathrm{Pic}_S^0(L)_p \to T_p(M_S)_\Gamma \to \widetilde{X_S}(L) \to 0.$$

In combination with (31), this concludes the proof of part (2). $\qquad\qquad\qquad\square$

Remark 4.6. (Ritter-Weiss modules and Tate sequences.) Assume that L is the top field in a finite, Galois extension L/K, of Galois group G and that $\mathbb{F}_q(t) \subseteq K \subseteq L$. Further, assume that the set S is G–equivariant. Then, all the $\mathbb{Z}_p$–modules involved in the proof of the above Proposition carry natural $\mathbb{Z}_p[G]$–module structures. Most importantly, due to their canonical constructions, all the exact sequences above are exact in the category of $\mathbb{Z}_p[G]$–modules.

Exact sequence (32) is the p–adic, function field analogue of the Ritter–Weiss exact sequence (see [15]), defining a certain extension class $\bigtriangledown_S$ of a module of S–divisors by an S–ideal class group, in the number field setting. This is what prompts the notation $\bigtriangledown_S(L)_p := T_p(M_S)_\Gamma$.

Further, since $T_p(M_S)$ is $\mathbb{Z}_p[G]$–projective, the exact sequence of $\mathbb{Z}_p[G]$–modules

$$(33) \qquad\qquad 0 \to U_S(L)_p \to T_p(M_S) \xrightarrow{1-\gamma} T_p(M_S) \to \bigtriangledown_S(L)_p \to 0,$$

is the p–adic, function field analogue of a Tate exact sequence (see [4] and also [15] for more details), in the case where S is not necessarily p–large.

Of course, in order to cement these analogies, one would have to compute the extension classes of (32) an (33) in $\mathrm{Ext}^1_{\mathbb{Z}_p[G]}(\widetilde{X}_S(L), \mathrm{Pic}_S^0(L)_p)$ and $\mathrm{Ext}^2_{\mathbb{Z}_p[G]}(\bigtriangledown_S(L)_p, U_S(L)_p)$ and show that they coincide with the class–field theoretically meaningful Ritter–Weiss and Tate classes, respectively. In [4], this was done ℓ–adically, for $\ell \neq p$, for the exact sequence (33), in the case where S is ℓ–large. (See Theorem 2.2 in loc.cit.) A proof of the p–adic analogue of that theorem (even in the case where S is p–large) is still missing in the literature, unless $|G|$ is not divisible by p, in which case this was proved in [4]. (See Theorem 2.2. in loc.cit.)

References

[1] B. Anglès, A. Bandini, F. Bars, and I. Longhi. Iwasawa main conjecture for the Carlitz cyclotomic extension and applications. *Math. Ann.*, 376(1-2):475–523, 2020.

[2] A. Bandini, F. Bars, and E. Coscelli. Fitting ideals of class groups in Carlitz-Hayes cyclotomic extensions. *J. Number Theory*, 232:261–282, 2022.

[3] E. Coscelli. Stickelberger Series and Iwasawa Main Conjectures for Function Fields, PhD Thesis, Milano, 2017.

[4] C. Greither and C. Popescu. Picard 1-motives and Tate sequences for function fields. *J. Théor. Nombres Bordeaux*, 33(3, part 1):835–852, 2021.

[5] C. Greither and C. D. Popescu. The Galois module structure of ℓ-adic realizations of Picard 1-motives and applications. *Int. Math. Res. Not. IMRN*, (5):986–1036, 2012.

[6] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[7] D. R. Hayes. Stickelberger elements in function fields. *Compositio Math.*, 55(2):209–239, 1985.

[8] K. Iwasawa. *Local class field theory*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986. Oxford Mathematical Monographs.

[9] J. A. Kiehlmann. Classifications of countably-based abelian profinite groups. *J. Group Theory*, 16(1):141–157, 2013.

[10] H. Kisilevsky. Multiplicative independence in function fields. *J. Number Theory*, 44(3):352–355, 1993.

[11] S. Lang. *Cyclotomic fields I and II*, volume 121 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990. With an appendix by Karl Rubin.

[12] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[13] H. Nishimura. On the unique factorization theorem for formal power series. *J. Math. Kyoto Univ.*, 7:151–160, 1967.

[14] C. D. Popescu. Integral and p-adic refinements of the abelian Stark conjecture. In *Arithmetic of L-functions*, volume 18 of *IAS/Park City Math. Ser.*, pages 45–101. Amer. Math. Soc., Providence, RI, 2011.

[15] J. Ritter and A. Weiss. A Tate sequence for global units. *Compositio Math.*, 102(2):147–178, 1996.

[16] M. Rosen. Formal Drinfeld modules. *J. Number Theory*, 103(2):234–256, 2003.

[17] J. Tate. *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$*, volume 47 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher.

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